Solutions to HW2

1. "Levinson's Theorem": A natural thing to do here is take the inner product of the equation with $u(t)$. That gives

$$\langle u, \partial_t u\rangle = \langle u, L(t)u\rangle + \langle u, F(t)\rangle. \tag{1}$$

Before going further I want to assume that $t \in I = (a,b)$ such that $u(t) \neq 0$ on I. Then on I we have $2\langle u, \partial_t u\rangle = 2||u||\partial_t(||u||)$ – this is $2\mathrm{Re}\{\langle u, (\partial_t u\rangle\} = 2||u||\partial_t||u||$ in a complex Hilbert space – and $2\langle u, L(t)u\rangle = \langle u, (L(t)+L^*(t))u\rangle$ – this is $2\mathrm{Re}\{\langle u, L(t)u\rangle\} = \langle u, (L(t)+L^*(t))u\rangle$ in a complex Hilbert space. Thus (1) implies

$$\begin{aligned} ||u||\partial_t(||u||) &= \langle u, \frac{1}{2}(L(t)+L^*(t))u\rangle + \langle u, F(t)\rangle \\ &\leq ||u||^2||(L(t)+L^*(t))/2||_{op} + ||F(t)||\cdot||u||, \end{aligned}$$

and we can divide by $||u$ to get

$$\partial_t||u|| \leq ||(L(t)+L^*(t))/2||_{op}||u|| + ||F(t)|| \tag{2}$$

Now the usual "integrating factor" argument applies: let $A_a(t) = \int_a^t ||(L(s)+L^*(s))/2||_{op}ds$. (2) implies

$$\partial_t(e^{-A_a(t)}||u(t)||) \leq e^{-A_a(t)}||F(t)|| \leq ||F(t)||. \tag{3}$$

Now, since $u(t)$ is continuously differentiable on (a,b) and continuous on $[a,b)$, integrating both sides of (3) gives

$$||u(t)|| \leq (||u(a)|| + \int_a^t ||F(s)||ds)e^{A_a(t)} \qquad (4).$$

for $t \in I$. Now, if $u(t)$ were never zero, we could take $I = (0, T_+)$. Unfortunately, because $F(t)$ can be any continuous vector-valued function, the set where $u(t) = 0$ can be an arbitrary closed subset of $\mathbb{R}$! However, this means that the set $||u(t)|| > 0$ is a countable disjoint union of intervals of the form $I = (a,b)$. If $a = 0$, then $u(a) = u_0$. If $a > 0$, then $u(a) = 0$. So on each of the intervals in $[0, T_+)$ where $||u(t)|| > 0$, the estimate (4) gives

$$||u(t)|| \leq (||u(a)|| + \int_a^t ||F(s)||ds)e^{A_a(t)} \leq (||u_0|| + \int_0^t ||F(s)||ds)e^{A_0(t)}. \tag{5}$$

Since (5) is always true when $||u(t)|| = 0$, it follows that (5) holds everywhere that the solution exists.

There is even a little difficulty in showing that the solution exists for $t \in [0,\infty)$. If you follow the standard procedure to make this an equation of the form $\dot{u} = G(u)$, i.e. introduce t as a new dependent variable and add the equation $\dot{t} = 1$, you will not get an equation where $G(u)$ is locally Lipschitz because $L(t)$ and $F(t)$ are only assumed to be continuous. The right way to proceed is to imitate Theorem 1.4 (Baby Picard) by considering

$$[\Phi(u)](t) = u(t_0) + \int_{t_0}^t (L(s)u(s) + F(s))ds$$

as a mapping on $C(\{|t-t_0| \leq \delta\} \to \mathcal{D})$. It is easy to see that for δ sufficiently small Φ is a strict contraction. Moreover , you will see that, given any N, you can use the same δ for $|t_0| \leq N$. Using that, you can patch together the solutions on small intervals – using uniqueness of solutions – to build a solution on the whole line.

2. The Bootstrap Problem. Here we are considering solutions of $\dot{u} = F(u)$ where F is locally Lipschitz and $M(r) = \max\{||F(u)||, ||u|| \leq r\}$. Let (T_-, T_+) be the interval of existence for the solution with $u(0) = u_0$. So for $T_0 < t < T_+$ we have

$$u(t) = u_0 + \int_0^t F(u(s))ds. \tag{6}$$

Note that as long as $||u(t)|| \leq r$ formula (6) implies

$$||u(t)|| \leq ||u_0|| + \int_0^t ||F(u(s))||ds \leq ||u_0|| + |t|M(r). \tag{7}$$

I want to show that this solution exists for $|t| < \delta$, where $\delta = (M(||u_0|| + 2))^{-1}$. I made that choice because (7) shows that if $||u(s)|| \leq ||u_0|| + 2$ for $0 \leq |s| \leq t < \min\{\delta, T_+, -T_-\}$ [call that hypothesis $H(t)$], then we actually have $||u(s)|| \leq ||u_0|| + 1$ for $0 \leq |s| \leq t < \min\{\delta, T_+, -T_-\}$ [call that conclusion $C(t)$]. So we have $H(t)$ implies $C(t)$ and $H(0)$ is clearly true. The continuity of $||u(t)||$ implies that the set where $C(t)$ is true is closed in $|t| < \min\{\delta, T_+, -T_-\}$ and that $H(s)$ holds for $|t-s| < \epsilon$ when $C(t)$ holds. So the Bootstrap Principle implies that $C(t)$ holds on $|t| < \min\{\delta, T_+, -T_-\}$. However, unless δ is strictly less than both T_+ and $-T_-$, this contradicts the fact that $||u(t)||$ blows up as t goes to T_+ and T_-. So the solution exists for $|t| < \delta$. There are informal ways of writing this argument that are a little shorter, but this is the way it looks following the Bootstrap Principle.

3. Exercise 1.35. This one is really easy ... except that E Poisson commuting with H does not make sense unless $E \in C^1_{loc}$. Please add that assumption. If $E(u)$ is an integral for $\partial_t u(t) = \nabla_\omega H(u(t))$, then by the formula $E(u(t)) = E(u(t_0)$ for all t for all solutions. Differentiating that with respect to t gives

$$0 = \langle \partial_t u(t), dE(u(t)) \rangle = \omega(\partial_t u(t), \nabla_\omega E(u(t))) = \omega(\nabla_\omega H(u(t)), \nabla_\omega E(u(t)))$$

Since $u(t)$ can be any point in $\mathcal{D}$, we have $0 = \omega(\nabla_\omega H(u), \nabla_\omega E(u))$ on $\mathcal{D}$.

For the opposite implication, if E and H Poisson commute we have

$$0 = \langle (\nabla_\omega H(u), dE(u) \rangle.$$

Thus for any differentiable curve $u(t)$ in $\mathcal{D}$,

$$\frac{d}{dt} E(u(t)) = \langle \partial_t u(t), dE \rangle = \langle \partial_t u(t) - \nabla_\omega H(u(t)), dE(u(t)) \rangle$$

Integrating both sides of this equality shows that $E(u)$ is an integral with $G(u) = dE(u)$.