Exercise 2.4: Group Velocity

This exercise is so relevant to the discussion of dispersive waves that I thought it was worth writing out. It is the basis for Principle 2.1 in the notes.

We are given $L = iP(D)$, where $p(\xi) = P(D)e^{ix\cdot\xi}$ is real-valued. Since $\widehat{e^{tL}f} = \exp(itp(\xi))\hat{f}$, it follows by the Plancherel theorem that $||e^{tL}f||_{L^2} = ||f||_{L^2}$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Since the Sobolev norm $H^s(\mathbb{R}^n)$ is given by

$$||f||_s^2 = \int \langle\xi\rangle^{2s}|\hat{f}(\xi)|^2 d\xi,$$

it also follows that $||e^{tL}f||_s = ||f||_s$ for all s, t and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proposition. Let $u_0^\epsilon(x) = \exp(ix\cdot\xi_0)\phi(\epsilon x),\ \phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$w^\epsilon(x,t) = e^{i(x\cdot\xi_0+tp(\xi_0))}\phi(\epsilon(x+t\partial_\xi p(\xi_0)))$$

satisfies

$$||w^\epsilon(\cdot,t) - e^{tL}u_0^\epsilon(\cdot)||_s \le C_{\xi_0}\epsilon^2|t|||h^\epsilon(\cdot)||_s, \tag{1}$$

where

$$h^\epsilon(x) = \sum_{j=2}^m \epsilon^{j-2}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha p(\xi_0)(D^\alpha\phi)(\epsilon x).$$

Note that for fixed ϕ we have $||h^\epsilon||_s/||u_0^\epsilon||_s$ bounded independently of ϵ so that w^ϵ really is a good approximation to $e^{tL}u_0^\epsilon$ for small ϵ on bounded intervals in time. So initial data with phase oscillation ξ_0 and nearly constant amplitude propagate with the true velocity $\vec{v} = -\partial_\xi p(\xi_0)$ which is called the "group velocity", while their phase increases at the rate $p(\xi_0)/|\xi_0|$ which is called the "phase velocity".

Proof: The proof of this is really what I did last Friday (02/04/06) with a few more observations. As in class one computes that

$$(\partial_t - L)w^\epsilon(x,t) = e^{i(x\cdot\xi_0+tp(\xi_0))}h(x+t\partial_\xi p(\xi_0)),$$

and applies Duhamel's formula to get a representation for $w^\epsilon - e^{tL}u_0^\epsilon$. Then one takes norms. Since $||\exp(ix\cdot\xi_0)f||_s \le C_{\xi_0}||f||_s$ and $||f(\cdot+v)||_s = ||f(\cdot)||_s$, (1) follows from the remarks on norms preceding the statement of the Proposition.