The Vitali Covering Lemma and an Application

These notes are based on portions of Chapter 5 of Royden's *Real Analysis*. Given a set $E \subset \mathbb{R}^d$, a Vitali covering of E is a set $\mathcal{B}$ of closed balls of positive radius such that every point in E is contained in balls in $\mathcal{B}$ of arbitrarily small radius. When you have such a cover, it is possible to choose *disjoint* balls from $\mathcal{B}$ which cover almost all of E:

Vitali's Covering Lemma: If $m_*(E) < \infty$, given $\epsilon > 0$, there are disjoint $B_j \in \mathcal{B},\ j = 1, .., N$, such that

$$m_*(E \cap (\cup B_j)^c) < \epsilon.$$

Note that by the subadditivity of outer measure this implies that $m_*(E \cap (\cup B_j)) > m_*(E) - \epsilon$, and hence the B_j's cover "most" of E.

Proof. First choose an open set $\mathcal{O}$ containing E such that $m(\mathcal{O}) < m_*(E) + 1$. Remove all balls from $\mathcal{B}$ that are not contained in $\mathcal{O}$. Note that the remainder of $\mathcal{B}$ is still a Vitali cover of E. Intuitively, one ought to be able to find disjoint $B_1, .., B_N$ covering most of E because any points of E not already in a finite set of $B_j \in \mathcal{B}$ are contained in balls disjoint from $\cup B_j$. The trick is finding a systematic way to choose the B_j's. Here is the procedure. For each n, choose $B_n \in \mathcal{B}$ disjoint from $\cup_{k=1}^{n-1} B_k$ such that the radius of B_n is greater than

$$\frac{1}{2} \sup\{\text{radii of } B \in \mathcal{B}, \text{ such that } B \cap (\cup_{k=1}^{n-1} B_k) = \emptyset\}$$

This supremum is not infinite because $B \in \mathcal{B}$ implies $B \subset \mathcal{O}$. If there is no $B \in \mathcal{B}$ such that $B \cap (\cup_{k=1}^{n-1} B_k) = \emptyset$, then $E \subset \cup_{k=1}^{n-1} B_k$, and we are done. So suppose that we have B_n for all n. Since the B_n's are disjoint and contained in $\mathcal{O}$,

$$\sum_{k=1}^{\infty} |B_k| \leq m(\mathcal{O}) < \infty, \text{ and we have } \sum_{k=N}^{\infty} |B_k| < \epsilon$$

for N sufficiently large. If $x \in E$ is not contained in $\cup_{k=1}^{\infty} B_k$, then there is a ball $B_0 \in \mathcal{B}$ such that $x \in B_0$, and, if $B_0 \cap B_k = \emptyset$ for $k = 1, .., l-1$, then the radius of B_0 must be less than twice the radius of B_l. Since the radii of the B_k's go to zero, there will be a *first* B_l which intersects B_0. If you multiply the radius of that B_l by 5, it will contain B_0 (draw a picture here). From this you conclude that the part of E not contained in $\cup_{k=1}^{N} B_k$, is contained in $\cup_{k=N+1}^{\infty} \tilde{B}_k$, where $\tilde{B}_k$ has the same center as B_k and 5 times the radius. Since $E \cap (\cup_{k=1}^{N} B_k)^c \subset \cup_{k=N+1}^{\infty} \tilde{B}_k$, we have

$$m_*(E \cap (\cup_{k=1}^{N} B_k)^c) < 5^d \epsilon,$$

and the proof is complete.

We can apply this covering lemma to generalize the theorem proven earlier on differentiability of real-valued functions.

Theorem 1 Suppose that $f(x)$ is a nondecreasing, real-valued function on $[a,b]$, $-\infty < a < b < \infty$. Then f is differentiable almost everywhere.

Remark. This proof is the same as the proof on pages 123-4 in Stein & Shakarchi, but one uses the Vitali Covering Lemma in place of the Rising Sun Lemma. When you combine Theorem 1 with the result that any function in $BV[a,b]$ can be written as the difference of two monotone functions, you obtain Theorem 3.4 (and eliminate Section 3.3).

Proof of Theorem 1. As before it will suffice to show that $m(E_{r,R}) = 0$ for all $r, R \in \mathbb{Q}$, where

$$E_{r,R} = \{x \in (a,b) : D^+ f(x) > R > r > D_- f(x)\}.$$

Suppose that $m_*(E_{r,R}) = s > 0$, and choose an open set $\mathcal{O} \subset (a,b)$ such that $E_{r,R} \subset O$ and $m(\mathcal{O}) < s+\epsilon$. For all $x \in E_{r,R}$ there are arbitrarily small $h > 0$ such that $f(x) - f(x-h) < rh$. The intervals $[x-h, x]$ in $\mathcal{O}$ where this holds form a Vitali cover of $E_{r,R}$. Use the covering lemma to get disjoint $I_k = [x_k - h_k, x_k],\ k = 1, .., N$, contained in $\mathcal{O}$ such that $f(x_k) - f(x_k - h_k) < rh_k$ and $m_*(E \cap (\cup_{k=1}^N I_k)) > s - \epsilon$. Note that $\sum_{k=1}^N h_k < s+\epsilon$, Letting $\dot{I}_k$ denote the interior of I_k, for all $x \in \dot{I}_k \cap E_{r,R}$ there are arbitrarily small $h > 0$ such that $f(x+h) - f(x) > Rh$. These again form a Vitali covering. So we can choose disjoint intervals $J_j = [x_j, x_j + h_j],\ j = 1, .., M$, such that each J_j is contained in one of the $\dot{I}_k$'s, $f(x_j + h_j) - f(x_j) > Rh_j$ and $m_*((E_{r,R} \cap (\cup J_j)) > s - 2\epsilon$. Now we can put this together to reach a contradiction to $s > 0$ essentially the same way as we did before: since f is monotone nondecreasing we have

$$R(s - 2\epsilon) < \sum_{j=1}^{M} f(x_j + h_j) - f(x_j) \leq \sum_{k=1}^{N} f(x_k) - f(x_k - h_k) < r(s + \epsilon).$$

Since ϵ is arbitrary and $r < R$, this is only possible if $s = 0$.