The Vitali Covering Lemma and an Application

These notes are based on portions of Chapter 5 of Royden’s *Real Analysis*. Given a set \( E \subset \mathbb{R}^d \), a Vitali covering of \( E \) is a set \( \mathcal{B} \) of closed balls of positive radius such that every point in \( E \) is contained in balls in \( \mathcal{B} \) of arbitrarily small radius. When you have such a cover, it is possible to choose disjoint balls from \( \mathcal{B} \) which cover almost all of \( E \):

**Vitali’s Covering Lemma:** If \( m_*(E) < \infty \), given \( \epsilon > 0 \), there are disjoint \( B_j \in \mathcal{B}, \ j = 1, \ldots, N \), such that

\[
m_*(E \cap (\bigcup B_j)^c) < \epsilon.
\]

Note that by the subadditivity of outer measure this implies that \( m_*(E \cap (\bigcup B_j)) > m_*(E) - \epsilon \), and hence the \( B_j \)'s cover “most” of \( E \).

**Proof.** First choose an open set \( \mathcal{O} \) containing \( E \) such that \( m(\mathcal{O}) < m_*(E) + 1 \). Remove all balls from \( \mathcal{B} \) that are not contained in \( \mathcal{O} \). Note that the remainder of \( \mathcal{B} \) is still a Vitali cover of \( E \). Intuitively, one ought to be able to find disjoint \( B_1, \ldots, B_N \) covering most of \( E \) because any points of \( E \) not already in a finite set of \( B_j \in \mathcal{B} \) are contained in balls disjoint from \( \bigcup B_j \). The trick is finding a systematic way to choose the \( B_j \)'s. Here is the procedure. For each \( n \), choose \( B_n \in \mathcal{B} \) disjoint from \( \bigcup_{k=1}^{n-1} B_k \) such that the radius of \( B_n \) is greater than

\[
\frac{1}{2} \sup \{ \text{radii of } B \in \mathcal{B}, \text{ such that } B \cap (\bigcup_{k=1}^{n-1} B_k) = \emptyset \}
\]

This supremum is not infinite because \( B \in \mathcal{B} \) implies \( B \subset \mathcal{O} \). If there is no \( B \in \mathcal{B} \) such that \( B \cap (\bigcup_{k=1}^{n-1} B_k) = \emptyset \), then \( E \subset \bigcup_{k=1}^{n-1} B_k \), and we are done. So suppose that we have \( B_n \) for all \( n \). Since the \( B_n \)'s are disjoint and contained in \( \mathcal{O} \),

\[
\sum_{k=1}^{\infty} |B_k| \leq m(\mathcal{O}) < \infty, \text{ and we have } \sum_{k=N}^{\infty} |B_k| < \epsilon
\]

for \( N \) sufficiently large. If \( x \in E \) is not contained in \( \bigcup_{k=1}^{\infty} B_k \), then there is a ball \( B_0 \in \mathcal{B} \) such that \( x \in B_0 \), and, if \( B_0 \cap B_k = \emptyset \) for \( k = 1, \ldots, l-1 \), then the radius of \( B_0 \) must be less than twice the radius of \( B_l \). Since the radii of the \( B_k \)'s go to zero, there will be a first \( B_l \) which intersects \( B_0 \). If you multiply the radius of that \( B_l \) by 5, it will contain \( B_0 \) (draw a picture here). From this you conclude that the part of \( E \) not contained in \( \bigcup_{k=1}^{N} B_k \), is contained in \( \bigcup_{k=N+1}^{\infty} \tilde{B}_k \), where \( \tilde{B}_k \) has the same center as \( B_k \) and 5 times the radius. Since \( E \cap (\bigcup_{k=1}^{N} B_k)^c \subset \bigcup_{k=N+1}^{\infty} \tilde{B}_k \), we have

\[
m_*(E \cap (\bigcup_{k=1}^{N} B_k)^c) < 5^d \epsilon,
\]

and the proof is complete.
We can apply this covering lemma to generalize the theorem proven earlier on differentiability of real-valued functions.

**Theorem 1** Suppose that \( f(x) \) is a nondecreasing, real-valued function on \([a, b]\), \(-\infty < a < b < \infty\). Then \( f \) is differentiable almost everywhere.

**Remark.** This proof is the same as the proof on pages 123-4 in Stein & Shakarchi, but one uses the Vitali Covering Lemma in place of the Rising Sun Lemma. When you combine Theorem 1 with the result that any function in \(BV[a, b]\) can be written as the difference of two monotone functions, you obtain Theorem 3.4 (and eliminate Section 3.3).

**Proof of Theorem 1.** As before it will suffice to show that \( m(E_{r, R}) = 0 \) for all \( r, R \in \mathbb{Q} \), where

\[
E_{r, R} = \{ x \in (a, b) : D^+ f(x) > R > r > D_- f(x) \}.
\]

Suppose that \( m_*(E_{r, R}) = s > 0 \), and choose an open set \( \mathcal{O} \subset (a, b) \) such that \( E_{r, R} \subset \mathcal{O} \) and \( m(\mathcal{O}) < s + \epsilon \). For all \( x \in E_{r, R} \) there are arbitrarily small \( h > 0 \) such that \( f(x) - f(x - h) < rh \). The intervals \([x-h, x]\) in \( \mathcal{O} \) where this holds form a Vitali cover of \( E_{r, R} \). Use the covering lemma to get disjoint \( I_k = [x_k - h_k, x_k], \ k = 1, \ldots, N \), contained in \( \mathcal{O} \) such that \( f(x_k) - f(x_k - h_k) < rh_k \) and \( m_*(E \cap (\cup_{k=1}^N I_k)) > s - \epsilon \).

Note that \( \sum_{k=1}^N h_k < s + \epsilon \). Letting \( \hat{I}_k \) denote the interior of \( I_k \), for all \( x \in \hat{I}_k \cap E_{r, R} \) there are arbitrarily small \( h > 0 \) such that \( f(x + h) - f(x) > Rh \). These again form a Vitali covering. So we can choose disjoint intervals \( J_j = [x_j, x_j + h_j], \ j = 1, \ldots, M \), such that each \( J_j \) is contained in one of the \( \hat{I}_k \)'s, \( f(x_j + h_j) - f(x_j) > Rh_j \) and \( m_*(E_{r, R} \cap (\cup J_j)) > s - 2\epsilon \). Now we can put this together to reach a contradiction to \( s > 0 \) essentially the same way as we did before: since \( f \) is monotone nondecreasing we have

\[
R(s - 2\epsilon) < \sum_{j=1}^M f(x_j + h_j) - f(x_j) \leq \sum_{k=1}^N f(x_k) - f(x_k - h_k) < r(s + \epsilon).
\]

Since \( \epsilon \) is arbitrary and \( r < R \), this is only possible if \( s = 0 \).