Assignment 8

Here is a collection of Rudin problems.

1. Assume that χ is an uncountable set, and let $\mathcal{M}$ be the set of all countable subsets of χ and their complements. Show that $\mathcal{M}$ is a σ-algebra and that μ, defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E^c is countable, is a measure on $\mathcal{M}$.

2. Suppose that $\mathcal{M}$ is a σ-algebra containing a countably infinite number of sets. Show that $\mathcal{M}$ cannot be countable.

3. A sequence of measurable functions $\{f_n\}$ *converges in measure* to a measurable function f with respect to μ if for every $\epsilon > 0$ there is an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

for $n \geq N$. Show that if $||f_n - f||_1 \to 0$ then f_n converges to f in measure. If $\mu(\chi) < \infty$ show that if $f_n(x)$ converges to $f(x)$ a.e., then f_n converges to f in measure, and conversely, if f_n converges to f in measure, then $\{f_n]$ has a subsequence that converges to f pointwise a.e.

4. If $\mu(\chi) = 1$, show that for any measurable function f and $p \in (0, \infty)$

$$\exp(\int_\chi \log|f| d\mu) \leq (\int_\chi |f|^p d\mu)^{1/p},$$

where $e^{-\infty}$ is defined to be 0. Somewhat surprisingly, if you assume that $\int_\chi |f|^r d\mu < \infty$ for some $r > 0$ and define $e^{-\infty} = 0$, the right hand side of that inequality converges to the left hand side as $p \to 0$. That is harder to show.

5. Suppose that f is a continuous function on the bounded interval $[a, b]$ with total variation $T_f[a, b] \leq \infty$. Show that for each $M < T_f[a, b]$ there is a $\delta > 0$ such that the variation of f over any partition of $[a, b]$ with mesh less that δ will be greater than M.