Assignment 5

There is a short proof of the Dominated Convergence Theorem directly from Fatou's Lemma: assuming that f_n converges pointwise a.e. to f, and $|f_n| \le g \in L^1(R^d)$, you can assume without loss of generality that the f_n's are real-valued and apply Fatou to both $g - f_n$ and $g + f_n$. I guarantee that when you do that you will get the Dominated Convergence Theorem. In fact, arguing this way you can do

1. (a) Assume that f_n, g_n, f and g are integrable, that f_n converges to f pointwise a.e. and g_n converges pointwise to g a.e., that $|f_n| \le g_n$ and that $\lim \int g_n = \int g$. Then $\lim \int f_n = \int f$. [This is the most general dominated convergence theorem that I know of.]

(b) (Quals) Given $f_n \in L^1$ converging pointwise to $f \in L^1$, show that $\lim \int |f_n - f|dx = 0$ if and only if $\lim \int |f_n|dx = \int |f|dx$.

2. (Quals) On [0,1] let $h_n(x) = \sum_{j=1}^{n} (-1)^j \chi_{j,n}(x)$ where $\chi_{j,n}$ is the characteristic function of $[(j-1)/n, j/n)$. Prove that $\lim_{n\to\infty} \int_0^1 f(x)h_n(x)dx = 0$ for all integrable f.

From the exercises for Chapter 3 in Stein & Shakarchi (pp. 145-152) **Exercises 4, 5, and 10**.