Solutions to Assignment 6

1. Applying l'Hôpital's rule in the variable h to this indeterminant form we have

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}.$$

Note that to apply l'Hôpital's rule we only need to check that the derivative of the denominator does not vanish for $h\neq 0$. Strictly speaking, you have to apply l'Hôpital twice in the form that Rudin states it: once on the interval $(a,b)=(0,\delta)$ where δ is small enough that the numerator is differentiable on this interval, and then once on the interval $(a,b)=(-\delta,0)$. Anyway then you have the equality above, and you can say

$$\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=\lim_{h\to 0}\frac{f'(x+h)-f'(x)-f'(x-h)+f'(x)}{2h}$$

$$=\frac{1}{2}[\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}+\lim_{h\to 0}\frac{f'(x-h)-f'(x)}{-h}]=f''(x).$$

2. The difficulty with a problem like this is deciding how much you have to say. Since Rudin (in section 5.4) points out that the product and sum rules imply $(x^n)'=nx^{n-1}$ for all n, you can certainly claim without proof that $|x|^3$ is differentiable to all orders for $x\neq 0$. Moreover, since $|x|^3=x^3$ for $x\geq 0$ and $|x|^3=-x^3$ for $x\leq 0$, you have for $x\geq 0$

$$\lim_{h\to 0_+}\frac{f(x+h)-f(x)}{h}=(x^3)'=3x^2,$$

and for $x\leq 0$

$$\lim_{h\to 0_-}\frac{f(x+h)-f(x)}{h}=(-x^3)'=-3x^2.$$

Since these limits are equal at $x=0$, $f'(0)$ exists and is zero. Repeating this argument, now that you know $(|x|^3)'=3x^2$ for $x\geq 0$ and $(|x|^3)'=-3x^2$ for $x\leq 0$, you will see $(|x|^3)''=6x$ for $x\geq 0$ and $(|x|^3)''=-6x$ for $x\leq 0$. Thus

$$\frac{f''(0+h)-f''(0)}{h}=6 \text{ for } h>0, \text{ but } \frac{f''(0+h)-f''(0)}{h}=-6 \text{ for } h<0$$

and the third derivative at zero is undefined.

3. Following Rudin's hint, you should rearrange

$$f(x+2h)=f(x)+2hf'(x)+2h^2f''(c) \text{ to get } f'(x)=\frac{1}{2h}[f(x+2h)-f(x)]-hf''(c)$$

Hence, for $h > 0$

$$|f'(x)| \leq= \frac{1}{2h}[|f(x+2h)| + |f(x)|] + h|f''(c)|.$$

Then, simply substituting the definitions of M_0 and M_2 gives

$$|f'(x)| \leq \frac{M_0}{h} + hM_2.$$

Now you just have to find the h which minimizes $M_0/h + hM_2$. Setting the derivative of $M_0/h + hM_2$ equal to zero and solving, you will find that the minimum occurs at $h = (M_0/M_2)^{1/2}$. Substituting this shows $|f'(x)| \leq 2(M_0 M_2)^{1/2}$. Since this holds for all $x \in (a, \infty)$, we have $M_1^2 \leq 4M_0M_2$.

I will assume that you were able to verify Rudin's example.

4. To show that $f'(x)$ goes to zero as $x \to \infty$, note that the hypothesis implies, given $\delta > 0$, there is an N such that $|f(x)| < \delta$ for $x > N$. Apply Rudin's problem 15 (i.e. problem 3 in this assignment) with $(a, \infty) = (N, \infty)$. This shows that, for $x > N$ $|f'(x)|^2 \leq 4\delta M_2$. Hence, given $\epsilon > 0$, if we take $\delta = \epsilon^2/(4M_2)$, we have $|f'(x)| < \epsilon$ for $x > N$, i.e. $\lim_{x\to\infty} f'(x) = 0$.

5. a) This is the MVT: if $f(x) = x$ and $f(y) = y$ where $x < y$, the MVT says there is a $c \in (x, y)$ such that $f(x) - f(y) = (x - y)f'(c)$, i.e. $x - y = (x - y)f'(c)$. So $f'(c) = 1$, contradicting the hypothesis $f'(t) \neq 1$.

b) $f(t) = t$ implies $(1 + e^t)^{-1} = 0$ which implies $1 = 0$. So $f(t)$ is never t. We have

$$f'(t) = 1 - e^t(1 + e^t)^{-2}.$$

Since $e^t > 0$, it follows that $(1 + e^t)^2 > (1 + e^t)$ and so $0 < e^t(1 + e^t)^{-2} < 1$. Thus $0 < f'(t) < 1$.

c) This is the interesting part. Take x_0 to be any point on the line, and define the sequence $\{x_n\}$ by $f(x_n) = x_{n+1}$. It will suffice to show that $\{x_n\}$ is a Cauchy sequence. If it is a Cauchy sequence in $\mathbb{R}$, it must converge to a point $x_\infty \in \mathbb{R}$, and, since f is continuous,

$$f(x_\infty) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_\infty.$$

To see that $\{x_n\}$ is a Cauchy sequence, one can argue as follows. If we ever have $x_{n+1} = x_n$, we are done, so we can assume that this does not happen. Then by the MVT

$$f(x_{n+1}) - f(x_n) = (x_n - x_{n-1})f'(c)$$

for some c between x_{n-1} and x_n. Thus, by hypothesis, $|x_{n+1} - x_n| \leq A|x_n - x_{n-1}|$ for all $n > 0$. Thus $|x_{n+1} - x_n| \leq A^n B$ with $B = |x_1 - x_0|$. Now the triangle inequality gives

$$|x_{n+m} - x_n| \leq \sum_{k=0}^{m-1} |x_{n+k+1} - x_{n+k}| \leq B \sum_{k=0}^{m-1} A^{n+k} \leq BA^n(1-A)^{-1}$$

by the formula for the sum of a geometric series. Since $A^n \to 0$ as $n \to \infty$, we can conclude that $\{x_n\}$ is a Cauchy sequence.