Some invariants of Poisson manifolds

by

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Abstract

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In this thesis we study various invariants of Poisson manifolds.

Given a compact oriented surface, we provide a complete classification of Poisson structures on it having at most quadratic degeneracies by constructing an explicit finite set of invariants. In the case that the Poisson tensor has at most linear degeneracies we also compute the Poisson cohomology, and explicitly describe (in terms of the invariants) deformations of the Poisson structures associated to various elements of the second cohomology.

We study the properties of gauge and Morita equivalence of Poisson manifolds. In particular, we show that gauge equivalent integrable Poisson manifolds are Morita equivalent. We prove that the leaf spaces of Morita equivalent Poisson manifolds are homeomorphic as topological spaces, and that the modular periods around the zero curves are invariant under Morita equivalence. As an example, we classify topologically stable Poisson structures on a two-sphere up to Morita equivalence and gauge equivalence.

We compute the Poisson cohomology of the standard r-matrix structure on the Poisson-Lie group $SU(2)$. In particular, the second cohomology turns out to be infinite-dimensional, which implies that there exist infinitely many linearly-independent infinitesimal deformations of the structure.

Professor Alan D. Weinstein
Dissertation Committee Chair

To my grandparents

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Chapter 1

Introduction

Poisson manifolds arose naturally in mathematical physics, in the course of studying physical systems possessing a symmetry. Recall that a *symplectic manifold M* (corresponding to a phase space in physics) provides the geometric framework for the Hamiltonian formulation of mechanics. A point in the symplectic manifold represents a state of the classical system. The presence of a symplectic form allows one to associate to each function H ("Hamiltonian" in physics) a vector field; the flow of this vector field gives the time evolution of the physical system described by the Hamiltonian. In the presence of symmetry of the physical system, it is natural to replace the manifold M by its quotient by the symmetry. The quotient manifold P no longer has a symplectic structure; but a trace of the symplectic structure remains. Namely, as was discovered by Poisson, the symplectic structure on M can be equivalently described by a Poisson bracket $\{\cdot,\cdot\}$ on the space of smooth functions on M. It is this structure that descends to the quotient manifold P. Thus it makes sense to consider Poisson manifolds (i.e., manifolds together with a Lie bracket $\{\cdot,\cdot\}$ satisfying the Leibniz identity on the space of smooth functions) as a generalization of symplectic manifolds. The Poisson bracket on a symplectic manifold is then characterized by the property of being locally isomorphic to the standard one on $\mathbb{R}^{2n}$. In a certain loose sense, every Poisson manifold is a quotient of a symplectic manifold by a symmetry.

If the Hamiltonian H on M is invariant under the symmetry, it descends to a function, $\tilde{H}$ on P. The equations of motion on P are then given by

$$\dot{F} = \{F, \tilde{H}\} \tag{1.0.1}$$

for any smooth function $F \in C^\infty(P)$ (here $\dot{F}$ refers to the time derivative of F). Note that F can be thought of as a "measurement" or an "observable" on the manifold P, assigning a number to each state of the physical system.

With the advent of quantum mechanics, it was noticed that in many instances there is a quantum analog of the equation above:

$$\dot{\hat{F}} = [\hat{F}, \hat{H}], \tag{1.0.2}$$

where $\hat{H}$ is the Hamiltonian of the quantum system, and $\hat{F}$ is an observable. The key difference in the quantum situation is that $\hat{F}$ and $\hat{H}$ are no longer functions, but operators on a Hilbert space, lying in a *non-commutative* algebra. The bracket $[\cdot,\cdot]$ is the commutator bracket for the operation of multiplication (composition) of operators.

This analogy motivated the study of *deformation quantization* [BFF+78]. The key idea is that a quantum-mechanical system must degenerate to a classical system, if considered in the range of energies in which quantum effects are insignificant (the so-called correspondence principle). Since one of the simplest examples of non-commutativity of quantum observables is the Heisenberg uncertainty principle

$$[\hat{P}, \hat{Q}] = i\hbar,$$

where $\hbar$ is Planck's constant, one should obtain classical mechanics from quantum mechanics by formally taking the limit $\hbar \to 0$.

Thus for each quantum observable one must have a corresponding classical observable. Since the multiplication of quantum observables is non-commutative, while the multiplication of classical observables is commutative, the multiplication $\times_\hbar$ must depend on the parameter $\hbar$. Mathematically, this means that the (non-commutative) algebras of quantum observables are defined by formal deformations of algebras of classical observables. This fact, and the correspondence between the evolution of the quantum system and the evolution of its classical limit gives a relation between the corresponding equations of motion (1.0.2) and (1.0.1):

$$\lim_{\hbar \to 0} [\hat{F}, \hat{H}]_\hbar = 0, \tag{1.0.3}$$

$$\lim_{\hbar \to 0} \frac{[\hat{F}, \hat{H}]_\hbar}{i\hbar} = \{F, \tilde{H}\}, \tag{1.0.4}$$

where $[\hat{F}, \hat{H}]_\hbar = \hat{F} \times_\hbar \hat{H} - \hat{H} \times_\hbar \hat{F}$. It follows from the recently proved Kontsevich's Formality Theorem [[Kon]] that every Poisson manifold admits a formal deformation quantization. That is, given a Poisson bracket $\{\,,\,\}$ on a manifold M, there exist a deformation of the multiplication on $C^\infty(M)$ so that the properties (1.0.3) and (1.0.4) are satisfied. Thus, Poisson manifolds can be viewed as *"semi-classical" limits* of phase spaces of quantum mechanical systems, or, equivalently, of a non-commutative algebras of quantum observables (see e.g. [CW99]). For this reason, Poisson manifolds have both a geometric and an algebraic side. Also, it is clear, since Poisson brackets arise as deformations of multiplication on an algebra, that the deformation theory (hence cohomology theory) of Poisson manifolds is of interest.

In this thesis, we are primarily interested in the question of classification of Poisson manifolds. Our goal is to find effective invariants that determine the possible Poisson structures on a given manifold, and to study infinitesimal deformations of these structures.

Because of the dual algebraic and geometric nature of Poisson manifolds and their relations with physics, there are many interesting notions of equivalence for Poisson structures.

Perhaps the most straightforward (and strongest) notion is that of a *Poisson isomorphism*. By that one means a diffeomorphism preserving Poisson brackets. It is in general a hopeless proposition to classify Poisson structures on a given manifold up to this notion of equivalence. Indeed, a Poisson structure induces a certain (singular) foliation of the underlying manifold, whose isomorphism class is clearly an invariant of the Poisson structure. Thus one would have to start by classifying all possible foliations of a certain type, which is a very hard problem.

Fortunately, the situation is much simpler in the case of a two-dimensional manifold. In Chapter 3, we give a complete set of invariants, allowing one to classify Poisson structures on a compact oriented surface, having at most quadratic degeneracies at a finite number of isolated points, and vanishing linearly on the rest of their zero sets. Besides the obvious invariant (the topology of the embedding of the zero set into the surface), there are numerical invariants of three kinds. The *modular eigenvalues* measure the rate of vanishing of the modular vector field at each quadratic degeneracy. The *modular flow times* measure the "energy" it takes to move between two points of quadratic degeneracy along an arc of a zero curve. (If there are no points of quadratic degeneracy on a zero curve, the corresponding invariant is the period of a modular vector field around the curve). The last invariant is the *regularized volume*, which is a certain generalization of the Liouville volume of a symplectic manifold.

When the Poisson tensor has at most linear degeneracies, we also compute the Poisson cohomology, and explicitly describe (in terms of the effect on our invariants) the infinitesimal deformations of the Poisson structures associated to the various cohomology elements.

Another, very recent, notion of equivalence for Poisson manifolds is that of *gauge equivalence*, introduced by P. Ševera and A. Weinstein. Their approach is to interpret Poisson structures as a particular case of the so-called Dirac structures on a manifold. Very roughly, the idea is to replace the Poisson tensor by the graph of the canonical map from the cotangent bundle to the tangent bundle that it defines. More generally, a Dirac structure gives rise to a certain subbundle of the direct sum of the cotangent and the tangent bundles. The additive group of differential two-forms on the manifold acts naturally on the space of all Dirac structures by "rotating" the associated graphs. Two structures are then called gauge-equivalent, if they belong to the same orbit of the action. It was shown by Ševera and Weinstein that gauge-equivalent Poisson structures are very "close" to each other (for example, they have the same cohomology); however, this equivalence relation remains rather mysterious.

Another, algebraically motivated, notion of equivalence of Poisson manifolds is that of *Morita equivalence*, introduced by P. Xu. His motivation was to start with the notion of representation equivalence (also called Morita equivalence) for algebras and arrive, by following the analogy between Poisson manifolds and non-commutative algebras, to the corresponding notion of Morita equivalence for Poisson manifolds.

In Chapter 4, we show that gauge equivalence of integrable Poisson manifolds implies their Morita equivalence, thus relating the two notions of equivalence. To obtain this result, we first prove that Poisson maps are equivariant with respect to gauge transformations of Poisson structures.

We also show that if two Poisson manifolds are Morita equivalent, their leaf spaces are homeomorphic; moreover, the modular periods of Morita equivalent Poisson structures around the corresponding zero curves (on which the Poisson structures vanish linearly) must be the same.

As an example, we classify topologically stable Poisson structures on a two-sphere up to Morita equivalence. To do so, we utilize our result that the modular periods around the zero curves and the topology of the leaf spaces are invariants for Morita equivalence. On the other hand, using the results on classification of structures with linear degeneracies obtained in Chapter 3, we show that if two topologically stable structures on a compact oriented surface have the same modular period invariants, but possibly different regularized volumes, they are gauge equivalent. It remains then to note that the topologically stable structures are integrable to conclude that, according to the main result of Chapter 4, they are Morita equivalent. The results in Chapter 4 are a part of the joint work ([BR]) with H. Bursztyn.

Finally, in Chapter 5, we compute the Poisson cohomology for the standard r-matrix Poisson-Lie structure on $SU(2)$. One of our results is that the second Poisson cohomology is infinite-dimensional. This means that the structure admits an infinite number of linearly-independent infinitesimal deformations.

Chapter 2

Preliminaries

We give here a short review of some basic notions in symplectic and Poisson geometry which will be used in the main part of the thesis. For a more detailed exposition, see, e.g., [Vai94, Wei98].

2.1 Symplectic manifolds

A *symplectic manifold* is a smooth (even-dimensional) manifold M with a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. This symplectic form gives rise to an invertible bundle map $\tilde{\omega} : TM \to T^*M$ according to

$$\tilde{\omega}(v)(u) = \omega(v,u), \qquad v,u \in TM$$

For a smooth real-valued function $f \in C^\infty(M)$ on a symplectic manifold (M,ω) the vector field $X_f \in \mathfrak{X}^1(M) = \Gamma(TM)$ given by

$$X_f = \tilde{\omega}^{-1}(df)$$

is called the *hamiltonian vector field of* f. The symplectic form defines a Lie bracket $\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on the space of smooth functions according to

$$\{f,g\} = X_g f = \omega(X_f,X_g), \qquad f,g \in C^\infty(M)$$

This bracket satisfies the *Leibniz identity*

$$\{f,gh\} = g\{f,h\} + \{f,g\}h, \qquad f,g,h \in C^\infty(M) \tag{2.1.1}$$

and is called the *symplectic Poisson bracket.*

The simplest example of a symplectic structure is given by $\mathbb{R}^{2n}$ with coordinates $(q_i, p_i)_{i=1}^n$ and the symplectic form

$$\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i. \tag{2.1.2}$$

A diffeomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ between two symplectic manifolds is called a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$. According to Darboux's theorem, the standard symplectic structure (2.1.2) provides a local model for any $2n$-dimensional symplectic manifold (M, ω): for any point $p \in M$ there exists a neighborhood U of p which is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

2.2 Poisson manifolds

2.2.1 Definition

A *Poisson structure* on a smooth manifold P is a Lie bracket on the space $C^\infty(P)$ which satisfies the Leibniz identity (2.1.1).

The Leibniz identity implies that the Poisson bracket $\{\,,\}$ is a derivation in each argument. Together with anti-symmetry of the bracket, this implies that there exists a bivector field $\pi \in \mathfrak{X}^2(P) = \Gamma(\Lambda^2 TP)$ such that

$$\{f, g\} = \langle \pi, df \wedge dg \rangle, \qquad f, g \in C^\infty(P),$$

where $\langle\,,\rangle : \Gamma(\Lambda^2 TP) \times \Gamma(\Lambda^2 T^*P) \to C^\infty(P)$ is the canonical pairing. The Jacobi identity for a Poisson bracket is equivalent to the following condition on the *Poisson bivector* π:

$$[\pi, \pi] = 0, \tag{2.2.1}$$

where $[\cdot, \cdot]$ denotes the Schouten bracket on the space $\mathfrak{X}^*(P)$ of multi-vector fields. Recall that the Schouten bracket $[\,,] : \mathfrak{X}^a(P) \times \mathfrak{X}^b(P) \to \mathfrak{X}^{a+b-1}(P)$ is the unique extension of the Lie bracket of vector fields and the action of vector fields on smooth functions such that

1. $[f, g] = 0, \qquad f, g \in C^\infty(P);$
2. $[X, f] = Xf, \qquad f \in C^\infty(P), X \in \mathfrak{X}^1(P);$
3. $[X, Y]$ is the commutator bracket of vector fields for $X, Y \in \mathfrak{X}^1(P)$;
4. $[A, B] = -(-1)^{a \cdot b}[B, A], \qquad A \in \mathfrak{X}^{a+1}(P), B \in \mathfrak{X}^{b+1}(P);$

5. For $A \in \mathfrak{X}^{a+1}(P)$, $[A,\cdot]$ is a derivation of degree a of the exterior multiplication on $\mathfrak{X}^*(P)$;

The signs in this definition are motivated by superalgebra. Another sign convention for the Schouten bracket is sometimes (e.g., in [Vai94]) used.

The Schouten bracket satisfies the graded Jacobi identity

$$[A,[B,C]] = [[A,B],C] + (-1)^{ab}[B,[A,C]] \tag{2.2.2}$$

for $A \in \mathfrak{X}^{a+1}(P)$, $B \in \mathfrak{X}^{b+1}(P)$ and $C \in \mathfrak{X}^{c+1}(P)$.

A Poisson structure π can be equivalently described in terms of a bundle map $\tilde{\pi} : T^*P \to TP$ such that

$$\alpha(\tilde{\pi}(\beta)) = \pi(\alpha,\beta), \qquad \alpha,\beta \in T^*P$$

For a smooth real-valued function $f \in C^\infty(P)$ the vector field $X_f = \tilde{\pi}(df)$ is called the Hamiltonian vector field of f.

When π has constant maximal rank (the corresponding bundle map $\tilde{\pi}$ is invertible and the manifold is even-dimensional in this case), it defines a symplectic structure on the manifold according to

$$\omega(X,Y) = \pi(\tilde{\pi}^{-1}(X), \tilde{\pi}^{-1}(Y)). \tag{2.2.3}$$

The non-degeneracy of the form (2.2.3) follows from the maximality of rank of π and the closedness of ω is equivalent to the integrability condition (2.2.1) on π. Conversely, a symplectic structure ω on a manifold P defines a Poisson structure by

$$\pi(\alpha,\beta) = \omega(\tilde{\omega}^{-1}(\alpha), \tilde{\omega}^{-1}(\beta)). \tag{2.2.4}$$

In this way, every symplectic manifold is an example of a Poisson manifold.

In general, π may have varying rank. The image of $\tilde{\pi}$ defines an involutive distribution. The Poisson structure on the manifold induces a symplectic structure on each of the integral manifolds of this distribution. The integral manifolds of this distribution are called the *symplectic leaves* of the Poisson manifold.

2.2.2 Local structure of a Poisson manifold.

It turns out that locally in a neighborhood of each point a Poisson manifold looks like a product of an open subset of the standard symplectic manifold $(\mathbb{R}^{2k}, \omega_0)$ for some $k \geq 0$ and a Poisson manifold whose Poisson bivector vanishes at the point of consideration. More precisely, the following theorem holds

Theorem. *(Splitting Theorem [Wei83]) Let (P,π) be a Poisson manifold, and let $x \in P$ be a point. Then there exist a neighborhood U of x with coordinates $((q_i, p_i)_{i=1}^k, (y_j)_{j=1}^l)$ such that on U we have*

$$\pi = \sum_{i=1}^{k} \partial_{q_i} \wedge \partial_{p_i} + \frac{1}{2} \sum_{i,j=1}^{l} \varphi_{ij}(y) \cdot \partial_{y_i} \wedge \partial_{y_j}, \qquad \varphi_{ij}(0) = 0.$$

When $l = 0$, the structure is symplectic, and the theorem reduces to the Darboux's theorem.

If a Poisson structure on P has constant rank on the neighborhood U, it is possible to choose coordinates in such a way that $\varphi_{ij}(y) \equiv 0$. If the rank of the Poisson structure on P is constant, the Poisson manifold (P,π) is called *regular*; in this case, the local decomposition of P is into the product of a symplectic manifold (identified with a piece of each leaf) and a local transversal to the symplectic leaves.

2.2.3 Example: Poisson structure on the dual of a Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra with a Lie bracket $[\,,\,]$, and $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Viewing $\mathfrak{g}$ as the subspace $\mathfrak{g} \subseteq \mathfrak{g}^{**} \subset C^\infty(\mathfrak{g}^*)$ consisting of linear functions, it is possible to extend the Lie bracket on $\mathfrak{g}$ to a Poisson bracket on $C^\infty(\mathfrak{g}^*)$, by first extending it to polynomials using the derivation property. Such an extension is unique, and the corresponding Poisson bracket is given by

$$\{f,g\}(\mu) = \langle \mu, [df(\mu), dg(\mu)] \rangle, \qquad f,g \in C^\infty(\mathfrak{g}^*),\, \mu \in \mathfrak{g}^*, \tag{2.2.5}$$

where $\langle\,,\,\rangle$ is the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. The Poisson structure (2.2.5) on the dual of a Lie algebra is often called the *Lie-Poisson structure*. Its symplectic leaves are the orbits of the coadjoint action.

2.3 Poisson cohomology

2.3.1 Definition

A Poisson structure π on a manifold P gives rise to a differential operator $d_\pi : \mathfrak{X}^*(P) \to \mathfrak{X}^{*+1}(P)$ of degree one on the space $\mathfrak{X}^*(P)$ of multivector fields on P. This operator was first introduced by Lichnerowicz [Lic77] and is given by Schouten bracket with π:

$$d_\pi X \doteq [\pi, X], \quad X \in \mathfrak{X}^*(P) \tag{2.3.1}$$

The condition $[\pi,\pi]=0$ together with the graded Jacobi identity for the Schouten bracket implies that $d_\pi^2=0$, making $(\mathfrak{X}^*(P),\, d_\pi)$ into a differential complex. The resulting cohomology

$$H^k_\pi(P) \doteq \frac{\ker(d_\pi : \mathfrak{X}^k(P) \to \mathfrak{X}^{k+1}(P))}{\mathrm{Im}(d_\pi : \mathfrak{X}^{k-1}(P) \to \mathfrak{X}^k(P))}$$

is called the *Poisson cohomology* of a Poisson manifold.

The map $\Omega^*(P) \to \mathfrak{X}^*(P)$ defined by the natural extension of $\tilde{\pi} : \Omega^1(P) \to \mathfrak{X}^1(P)$ according to

$$\alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \tilde{\pi}(\alpha_1) \wedge \cdots \wedge \tilde{\pi}(\alpha_n), \qquad \alpha_1, \ldots, \alpha_n \in \Omega^1(P)$$

is a morphism from the de Rham complex $(\Omega^*(P), d_{\mathrm{deRham}})$ to the Poisson complex $(\mathfrak{X}^*(P), d_\pi)$.

Example 2.3.1. Let (M,ω) be a symplectic manifold. Since $\tilde{\pi} = \tilde{\omega}^{-1} : T^*M \to TM$ is an isomorphism of vector bundles, the morphism $(\Omega^*(M),\, d_{\mathrm{deRham}}) \to (\mathfrak{X}^*(M),\, d_\pi)$ is an isomorphism of complexes. Therefore, the Poisson cohomology of a symplectic manifold is isomorphic to its de Rham cohomology: $H^*_\pi(M) \simeq H^*_{\mathrm{deRham}}(M)$.

Example 2.3.2. Let (P,π) be a Poisson manifold with the zero Poisson structure, $\pi \equiv 0$. Then $d_\pi = 0$ and the cohomology spaces $H^k_\pi(P) \simeq \mathfrak{X}^k_\pi(P)$ for $k = 0, \ldots, \dim(P)$ are infinite-dimensional.

In general, the Poisson cohomology combines the properties of the two extreme examples above.

Example 2.3.3. Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$ with its Lie-Poisson structure described in Example 2.2.3. Then, by a result of J.-H.Lu [Lu91], $H^*_\pi(\mathfrak{g}^*) \simeq H^*_{\mathrm{Lie}}(\mathfrak{g}, C^\infty(\mathfrak{g}^*))$, where the right hand side is the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $C^\infty(\mathfrak{g}^*)$. This isomorphism of cohomology spaces comes from an explicit isomorphism of the corresponding complexes.

2.3.2 Interpretations of cohomology spaces.

Let (P,π) be a Poisson manifold. The Poisson cohomology in low degrees has the following interpretations:

1. In degree 0, the differential d_π assigns to each function $f \in C^\infty(P)$ its Hamiltonian vector field: $d_\pi f = [\pi, f] = X_f$. Then $d_\pi f = 0$ iff $X_f g = \{f, g\} = 0$ for all $g \in C^\infty(P)$. Hence, $H^0_\pi(P)$ is the space of so-called *Casimir functions*, i.e. those functions which commute with all smooth functions with respect to the Poisson bracket.

2. In degree 1, for a vector field $X \in \mathfrak{X}^1(P)$ we have $d_\pi X = -L_X\pi$. Hence, $H^1_\pi(P)$ is the quotient of the space of so-called *Poisson vector fields* (whose flow preserves π) by the subspace of Hamiltonian vector fields. Interpreting Poisson vector fields as infinitesimal automorphisms, and Hamiltonian vector fields as infinitesimal inner automorphisms, one can think of $H^1_\pi(P)$ as of the space of *outer automorphisms* of (P,π);

3. To find an interpretation of $H^2_\pi(P)$, consider a formal one-parameter deformation of a Poisson structure π given by
$$\pi(\varepsilon) = \pi + \varepsilon \cdot \pi_1 + \varepsilon^2 \cdot \pi_2 + \dots, \tag{2.3.2}$$
where $\pi_i \in \mathfrak{X}^2(P)$, $i \geq 1$ and ε is a formal parameter. The condition for $\pi(\varepsilon)$ to be a Poisson bivector gives
$$[\pi(\varepsilon),\pi(\varepsilon)] = [\pi,\pi] + 2\varepsilon \cdot [\pi,\pi_1] + \varepsilon^2 \cdot (2[\pi,\pi_2] + [\pi_1,\pi_1]) + \dots = 0. \tag{2.3.3}$$
Since π is a Poisson structure, $[\pi,\pi] = 0$. If $d_\pi\pi_1 = [\pi,\pi_1] = 0$, then $[\pi+\varepsilon\pi_1,\pi+\varepsilon\pi_1] = O(\varepsilon^2)$ and $\pi_1 \in \mathfrak{X}^1(P)$ is called an *infinitesimal deformation* of π. If $\pi_1 = d_\pi X = -L_X\pi$ for some $X \in \mathfrak{X}^1(P)$, then $\pi + \varepsilon \cdot \pi_1$ is a Poisson bivector and π_1 is called a *trivial infinitesimal deformation* of π. Therefore, $H^2_\pi(P)$ is the space of infinitesimal deformations of π modulo its trivial infinitesimal deformations.

4. To find an interpretation of $H^3_\pi(P)$, we return to the equation (2.3.3). Suppose that π_1 is an infinitesimal deformation of π, i.e. $[\pi,\pi_1] = 0$. The coefficient of ε^2 in (2.3.3) is zero iff
$$d_\pi\pi_2 = -\frac{1}{2}[\pi_1,\pi_1]. \tag{2.3.4}$$
The graded Jacobi identity together with $[\pi,\pi] = 0$, $[\pi,\pi_1] = 0$ implies that $d_\pi[\pi_1,\pi_1] = 0$. Therefore, $[\pi_1,\pi_1]$ determines a class in $H^2_\pi(P)$. This class is zero iff (2.3.4) has a solution with respect to π_2. In general, the recursive solution of (2.3.3) involves at the n-th step solving an equation of the form
$$d_\pi\pi_n = \text{quadratic expression in } \pi_1,\dots,\pi_{n-1}. \tag{2.3.5}$$
Therefore, $H^3_\pi(P)$ contains the obstructions to extensions of infinitesimal deformations to formal deformations of higher orders.

2.3.3 The Mayer-Vietoris Sequence.

One of the tools useful in computations of Poisson cohomology is the *Mayer-Vietoris exact sequence* (see, e.g. [Vai94]). Its existence follows from the fact that d_π is functorial with respect to restrictions to open subsets. Explicitly, for a Poisson manifold (P,π) and open subsets $U,V \subset P$ the short exact sequence

$$0 \to \mathfrak{X}^*(U \cup V) \to \mathfrak{X}^*(U) \oplus \mathfrak{X}^*(V) \to \mathfrak{X}^*(U \cap V) \to 0$$

leads to a long exact sequence in cohomology:

$$\cdots \to H_\pi^{*-1}(U \cap V) \to H_\pi^*(U \cup V) \to H_\pi^*(U) \oplus H_\pi^*(V) \to H_\pi^*(U \cap V) \to \dots$$

2.3.4 Algebraic structures on $H_\pi^*(P)$

The cohomology space $H_\pi^*(P)$ has the structure of an associative graded commutative algebra and (after the necessary shifting of the degrees) the structure of a graded Lie algebra, which are obtained in the following way.

- Since $d_\pi(X \wedge Y) = d_\pi X \wedge Y + (-1)^{\deg X} X \wedge d_\pi Y$, the wedge product $\wedge : \mathfrak{X}^k(P) \times \mathfrak{X}^l(P) \to \mathfrak{X}^{k+l}(P)$ induces an associative graded commutative multiplication (or cup product) on Poisson cohomology: $[X] \wedge [Y] = [X \wedge Y]$, for $[X], [Y] \in H_\pi^*(P)$.
- Since $d_\pi([X,Y]) = -[d_\pi X, Y] - (-1)^{\deg X}[X, d_\pi Y]$, the Schouten bracket of multivector fields induces the bracket on $H_\pi^*(P)$: $[[X],[Y]] = [[X,Y]]$ for $[X],[Y] \in H_\pi^*(P)$, which becomes graded anti-commutative and satisfies the graded Jacobi identity if the degrees of the elements are shifted by -1.

2.4 Modular vector fields and the modular class of a Poisson manifold

The modular flow of a Poisson manifold is a one-parameter group of automorphisms determined by the choice of a smooth density on the manifold. The modular automorphism group of a von Neumann algebra A is a 1-parameter group of automorphisms of A, whose class modulo inner automorphisms is canonically associated to A. Considering Poisson manifolds as "semiclassical limits" of von Neumann algebras, A. Weinstein (see [Wei97] and references therein) arrived at the following definition of the modular vector field of a Poisson manifold.

Let (P,π) be a Poisson manifold and μ be a smooth positive density on P. Associated to this data, there is an operator $\varphi^\mu : f \mapsto \mathrm{div}_\mu X_f$ on the space $C^\infty(P)$ of smooth functions on P.

The anti-symmetry of the Poisson tensor π implies that φ^μ is a derivation of $C^\infty(P)$ and, therefore, defines a vector field on P. This vector field is called the *modular vector field* of P with respect to μ and is denoted by X^μ:

$$X^\mu f \doteq \mathrm{div}_\mu X_f = \frac{L_{X_f}\mu}{\mu}, \quad f \in C^\infty(P)$$

The modular vector field has the following properties:

1. $X^\mu = 0$ iff μ is invariant under the flows of all hamiltonian vector fields of π;
2. For any other $\mu' = a \cdot \mu$, $a \in C^\infty(P)$ the difference $X^{\mu'} - X^\mu$ is the hamiltonian vector field $X_{-\log a}$;
3. The flow of X^μ preserves π and μ : $L_{X^\mu}\pi = 0$, $L_{X^\mu}\mu = 0$;
4. X^μ is tangent to the symplectic leaves of maximal dimension.

If $X^\mu = 0$, μ is called an *invariant density*, and the Poisson manifold is called *unimodular.* For example, all symplectic manifolds are unimodular since the Liouville density associated to the symplectic structure is invariant under all hamiltonian flows (and the corresponding modular vector field is zero).

Since a modular vector field preserves the Poisson structure and depends on μ up a hamiltonian vector field, it determines a canonical class in the first Poisson cohomology, called the *modular class* of a Poisson manifold.

Example 2.4.1. For a Poisson structure on $\mathbb{R}^2$ given by $\pi = f(x,y)\partial_x \wedge \partial_y$, the modular vector field of π with respect to the density $\mu = |dx \wedge dy|$ is the same as the hamiltonian vector field of f with respect to the canonical non-degenerate Poisson structure $\pi_0 = \partial_x \wedge \partial_y$. Indeed, for any h,

$$X^\mu h = \frac{L_{f(x,y)(\partial_y h \partial_x - \partial_x h \partial_y)}(dx \wedge dy)}{dx \wedge dy} = \partial_y f \partial_x h - \partial_x f \partial_y h = \{h, f\}_{\pi_0} = X_f^{\pi_0} h.$$

2.5 Symplectic groupoids and Morita equivalence of Poisson manifolds

2.5.1 Groupoids

A *groupoid* over a set Γ_0 is a set Γ together with the following maps:

1. A pair of surjective maps $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$. The map α is called the *source* map, and β is called the *target* map of the groupoid.

2. A *product* $m : \Gamma^{(2)} \to \Gamma$ (also denoted by $m(g,h) = g \cdot h$) defined on the *set of composable pairs*:
$$\Gamma^{(2)} \doteq \{(g,h) \in \Gamma \times \Gamma | \, \beta(g) = \alpha(h)\}.$$
This product (multiplication) must satisfy the following properties:

   - For all $(g,h) \in \Gamma^{(2)}$, $\alpha(g \cdot h) = \alpha(g), \quad \beta(g \cdot h) = \beta(h);$
   - For all $(g,h),(h,k) \in \Gamma^{(2)}$, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ (associativity);

3. An embedding $\varepsilon : \Gamma_0 \to \Gamma$ called the *identity section* such that
$$\varepsilon(\alpha(g)) \cdot g = g = g \cdot \varepsilon(\beta(g)).$$

4. An *inversion* map $\iota : \Gamma \to \Gamma$ (also denoted by $\iota(g) = g^{-1}$) such that
$$\iota(g) \cdot g = \varepsilon(\beta(g)), \quad g \cdot \iota(g) = \varepsilon(\alpha(g)).$$

An element $g \in \Gamma$ can be thought of as an arrow from $x = \alpha(g) \in \Gamma_0$ to $y = \beta(g) \in \Gamma_0$.

Example 2.5.1. Any group G is a groupoid over its identity element $e \in G$.

Example 2.5.2. For a set Γ_0, the *pair groupoid* over Γ_0 is $\Gamma = \Gamma_0 \times \Gamma_0$ with the following structure maps:
$$\begin{aligned}
&\alpha(x,y) = x, \quad \beta(x,y) = y; \\
&(x,y) \cdot (y,z) = (x,z); \\
&\varepsilon(x) = (x,x); \\
&(x,y)^{-1} = (y,x);
\end{aligned}$$
for all $x,y,z \in \Gamma_0$.

2.5.2 Lie groupoids and Lie algebroids

A *Lie groupoid* Γ over a manifold Γ_0 is a groupoid over Γ_0 which has a structure of a smooth manifold such that

1. α, β are smooth submersions;

2. m, ε, ι are smooth maps;

For example, a Lie group is a Lie groupoid over its unit element. To any Lie group, one can associated an infinitesimal object, its *Lie algebra*. In a similar way, every Lie groupoid gives rise to an infinitesimal object: a *Lie algebroid.*

A *Lie algebroid* over a manifold P is a vector bundle $A \to P$ together with a Lie algebra structure $[\cdot,\cdot]$ on the space $\Gamma(A)$ of sections of A and a bundle map (called an *anchor*) $\rho : A \to TP$ such that

1. The induced map $\rho : \Gamma(A) \to \mathfrak{X}^1(P)$ is a Lie algebra homomorphism;

2. For any $f \in C^\infty(P)$ and $v, w \in \Gamma(A)$ the following Leibniz identity holds:

$$[v, fw]_A = f[v,w]_A + (\rho(v) \cdot f)w.$$

In contrast to the situation for Lie algebras (for which there is always a Lie group, "integrating" the given Lie algebra), there are Lie algebroids which do not come from any Lie groupoids. Those Lie algebroids for which there is a Lie groupoid "integrating" them are called *integrable*.

For any manifold M, its tangent bundle TM has a standard structure of a Lie algebroid: the bracket on the sections of TM is just the commutator bracket of vector fields and the anchor is the identity map. This algebroid is always integrable: as an integrating groupoid one can take, for example, the pair groupoid $M \times M$.

Every Lie groupoid has two associated anti-isomorphic Lie algebroid structures canonically defined on the normal bundle to its unit submanifold.

2.5.3 The Lie algebroid and the symplectic groupoid of a Poisson manifold

A Poisson structure π on a manifold P defines a Lie algebroid structure on the cotangent bundle T^*P in the following way. The Lie bracket on $\Omega^1(P) = \Gamma(T^*P)$ is determined be the condition that on the exact forms it is given by

$$[df, dg] = d\{f,g\}, \quad f,g \in C^\infty(P)$$

and it is extended to all 1-forms using the Leibniz identity

$$[\alpha, f\beta] = f[\alpha,\beta] + (-\tilde{\pi}(\alpha)\cdot f)\beta \quad \alpha,\beta \in \Omega^1(P), f \in C^\infty(P).$$

The anchor is given by $\rho \doteq -\tilde{\pi}$. With these definitions, the bundle map $\tilde{\pi} : \Omega^1(P) \to \mathfrak{X}^1(P)$ is a Lie algebra anti-homomorphism. This Lie algebroid is called the *Lie algebroid of a Poisson manifold.*

If there is a Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} P$ whose corresponding Lie algebroid is isomorphic to $(T^*P, [\,,\,], \tilde{\pi})$, the Poisson manifold P is called *integrable*. When the α-fibers (i.e., the fibers of the map $\alpha : \Gamma \to P$) are connected and simply-connected, the canonical symplectic structure on the cotangent bundle T^*P induces a symplectic structure Ω on Γ for which the graph $\{(z,x,y) \in \Gamma\times\Gamma\times\Gamma : z = x\cdot y\}$ of the groupoid multiplication is a lagrangian submanifold of $(\Gamma,\Omega)\times(\Gamma,-\Omega)\times(\Gamma,-\Omega)$. In this case $(\Gamma,\Omega,\alpha,\beta)$ is called a *symplectic groupoid* of the Poisson manifold. The source map $\alpha : \Gamma \to P$ of a symplectic groupoid is a Poisson map, and the target map $\beta : \Gamma \to P$ is an anti-Poisson map. Conversely, a symplectic structure on a groupoid which is compatible with the groupoid multiplication (i.e., the graph of the multiplication is lagrangian) induces a Poisson structure on the base of the groupoid so that the source map is a Poisson map.

A *symplectic realization* of a Poisson manifold (P,π) is a Poisson map φ from a symplectic manifold (Q,Ω) to (P,π). For example, the source map α of a symplectic groupoid of a Poisson manifold gives its symplectic realization. According to a theorem of Karasev [Kar87] and Weinstein [Wei83], every Poisson manifold has a surjective submersive symplectic realization. Symplectic realizations of Poisson manifolds can be considered as an analog of representations of associative algebras.

2.5.4 Morita equivalence of Poisson manifolds

Definition 2.5.3. ([Xu91a]) Two Poisson manifolds (P_1,π_1) and (P_2,π_2) are called *Morita equivalent* if there is a symplectic manifold (S,Ω) and surjective submersions $\alpha : S \to P_1$ and $\beta : S \to P_2$ such that

- $\alpha : (S,\Omega) \to (P_1,\pi_1)$ is a Poisson map;
- $\beta : (S,\Omega) \to (P_2,\pi_2)$ is an anti-Poisson map;
- α and β are complete maps of constant rank;
- α and β have connected simply connected fibers;

- the fibers of α and β are symplectic orthogonal to each other, $\ker(T\alpha) = \ker(T\beta)^{\perp}$.

Such a symplectic manifold (S,Ω) is called a *Morita equivalence bimodule* of (P_1,π_1), (P_2,π_2). The last property, in particular, implies that $\{\alpha^*(C^\infty(P_1)),\beta^*(C^\infty(P_2))\} = 0$. Equivalently, this means that the map $S \overset{\alpha\times\beta}{\to} P_1 \times P_2$ is a Poisson map, where $P_1 \times P_2$ is endowed with the product Poisson structure. An important property of Morita-equivalent Poisson manifolds is stated in the following

Proposition 2.5.4. *(see, e.g., [CW99]) There is a one-to-one correspondence of the leaves of Morita equivalent Poisson manifolds.*

Example 2.5.5. Let P be a connected and simply connected symplectic manifold, and M be a connected manifold with the zero Poisson structure. Then the manifold $\tilde{M} = P \times M$ with the product Poisson structure is Morita-equivalent to M, with the Morita equivalence bimodule given by $S = P \times T^*M$, and $\alpha = (\mathrm{id},\mathrm{pr}) : S \to P \times M$, $\beta = \mathrm{pr} : S \to M$. In particular, a connected and simply connected symplectic manifold is Morita equivalent to a point with the zero Poisson structure.

For integrable Poisson manifolds, Xu [Xu91a] showed that Morita equivalence for Poisson manifolds is the natural notion of equivalence in the category of symplectic groupoids. Therefore, one gets that for integrable Poisson manifolds, Morita equivalence is a true equivalence relation (in particular, it is transitive).

Chapter 3

Poisson structures on compact oriented surfaces.

3.1 Summary of results.

Recently several results were obtained concerning the *local classification* of Poisson structures on a manifold. According to the Splitting Theorem (Theorem 2.2.2), the problem of local classification of Poisson structures can be reduced to the classification of structures vanishing at a point. In dimension 2, V. Arnold [Arn78] obtained a hierarchy of normal forms of germs of Poisson structures degenerate at a point (see also P.Monnier [Mon] for a detailed exposition). Using the notion of the modular vector field of a Poisson structure, J.-P. Dufour and A. Haraki [DH91] and Z.-J. Liu and P. Xu [LX92] obtained a complete local classification of quadratic Poisson structures in dimension 3. Some results related to local classification of Poisson structures in dimensions 3 and 4 were also obtained by J. Grabowski, G. Marmo and A.M. Perelomov in [GMP93].

However, not much is known in relation to the *global classification* of Poisson structures on a given manifold (i.e., classification up to a Poisson isomorphism, see [Wei98] for a general discussion). In this chapter we prove several global classification results pertaining to certain types of Poisson structures on surfaces. Even though we always consider an equivalence via an orientation-preserving Poisson isomorphism, we note that the question of global equivalence by orientation-reversing Poisson isomorphisms can be reduced to the orientation-preserving case in the following way. Let $\nu : \Sigma \to \Sigma$ be an orientation-reversing diffeomorphism of a compact connected oriented surface Σ. Two Poisson structures π, π' on Σ are globally equivalent via an orientation-reversing diffeomorphism iff $\nu_*\pi$ and π' are globally equivalent via an orientation-preserving diffeomorphism.

In Section 3.2 we consider the Poisson structures whose degeneracies are at most linear (the simplest structures in Arnold's local classification). We call such structures *topologically stable*, since the topology of their zero set is preserved under small perturbations. The set of such structures is dense in the vector space of all Poisson structures on a given surface. The main result (Theorem 3.2.13) of Section 3.2 is a complete classification of these topologically stable structures up to an orientation-preserving Poisson isomorphism. In Section 3.2.7 we compute the Poisson cohomology of a given topologically stable Poisson structure vanishing linearly on n disjoint smooth curves on a compact oriented surface Σ of genus g. The zeroth cohomology (interpreted as the space of Casimir functions) is generated by constant functions and is one dimensional. The first cohomology (interpreted as the space of Poisson vector fields modulo Hamiltonian vector fields) has dimension $2g+n$ and is generated by the image of the first de Rham cohomology of Σ under the injective homomorphism $\tilde{\pi} : H^*_{\text{deRham}}(\Sigma) \to H^*_\pi(\Sigma)$ and by the following n vector fields:

$$X^{\omega_0}(\pi) \cdot \{\text{bump function around } \gamma_i\}, \quad i = 1, \ldots n.$$

The second cohomology is generated by a non-degenerate Poisson structure π_0 on Σ and n Poisson structures of the form

$$\pi_i = \pi \cdot \{\text{bump function around } \gamma_i\}, \quad i = 1, \ldots, n.$$

Each of the generators of the second cohomology corresponds to a one-parameter family of infinitesimal deformations of the Poisson structure which affects exactly one of the numerical classifying invariants. The deformation $\pi \mapsto \pi + \varepsilon \cdot \pi_0$ changes the regularized Liouville volume. For each $i = 1, \ldots, n$, the deformation $\pi \mapsto \pi + \varepsilon \cdot \pi_i$ changes the modular period around the curve γ_i. This shows that the number of numerical classifying invariants $(n+1)$ for $\mathcal{G}_n(\Sigma))$ equals the dimension of the second Poisson cohomology, and is, therefore, optimal.

As an example, we consider the classification of topologically stable Poisson structures on the sphere (Section 3.2.8) and describe the moduli space of such structures up to Poisson isomorphisms.

In Section 3.3, we explain how our techniques can be used to provide effective classification results for Poisson structures with zeros of higher order. In particular, we give a complete classification of Poisson structures π which vanish linearly on nearly all points in their zero set, except possibly having quadratic degeneracies at a finite number of points. As an example, we consider the structures on two-torus defined by a height function.

3.2 A classification of topologically stable Poisson structures on a compact oriented surface.

In this section we will find a complete classification of Poisson structures vanishing at most linearly on a given compact oriented surface. Locally these structures are the simplest in the Arnold's hierarchy.

3.2.1 A classification of symplectic structures

According to Darboux's theorem, all symplectic structures on a given manifold M are locally equivalent: for a symplectic form ω and a point $p \in M$ there exist a coordinate system $(\mathscr{U}, x_1, \cdots, x_n, y_1, \cdots, y_n)$ centered at p such that $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathscr{U}$. Therefore, the dimension of the manifold is the only local invariant of a symplectic structure.

Definition 3.2.1. Two symplectic forms ω_0 and ω_1 on M are *globally equivalent* if there is a symplectomorphism $\varphi : (M, \omega_0) \to (M, \omega_1)$.

In certain cases the following theorem of Moser allows one to classify symplectic forms on a manifold up to global equivalence:

Theorem 3.2.2. *(Moser, [Mos65]) Let ω_0 and ω_1 be symplectic forms on a compact $2n$ dimensional manifold M. Suppose that $[\omega_0] = [\omega_1] \in H^2_{deRham}(M)$ and that the 2-form $\omega_t \doteq (1-t)\omega_0 + t\omega_1$ is symplectic for each $t \in [0,1]$. Then there is a symplectomorphism $\varphi : (M, \omega_0) \to (M, \omega_1)$.*

The total Liouville volume $V(\omega) \doteq \int_M \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$ associated to a symplectic structure ω on M is a global invariant. That is, if symplectic forms ω_1 and ω_2 on M are globally equivalent, their Liouville volumes are equal, $V(\omega_1) = V(\omega_2)$.

In the case of a compact 2-dimensional manifold Moser's theorem implies

Corollary 3.2.3. *On a compact connected surface Σ two symplectic structures ω_0, ω_1 are globally equivalent iff the associated Liouville volumes are equal: $\int_\Sigma \omega_0 = \int_\Sigma \omega_1$.*

For completeness, we sketch a proof of the corollary (the proof of Moser's theorem is essentially the same). The idea is to find a time-dependent vector field X_t, whose flow at time $t = 1$ would take ω_0 to ω_1. Since $\int_D \omega_1 = \int_D \omega_2$, the class of $\omega_0 - \omega_1$ in the second de Rham cohomology is trivial. Hence $\omega_0 - \omega_1 = d\mu$ for a 1-form μ. Then for $X_t \doteq -\tilde{\omega}_t^{-1}(\mu)$ we have

$$L_{X_t}\omega_t = d\imath_{X_t}\omega_t + \imath_{X_t} d\omega_t = d\imath_{X_t}\omega_t = -(\omega_0 - \omega_1)$$

Therefore,

$$L_{X_t}\omega_t + \frac{d\omega_t}{dt} = 0.$$

Let ρ_t be the flow of the time-dependent vector field X_t. Since

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(L_{X_t}\omega_t + \frac{d\omega_t}{dt}\right) = 0,$$

$\rho_t^*\omega_t$ is constant and hence equal to $\rho_0^*\omega_0 = \omega_0$ for all $t \in [0,1]$. Thus in particular $\rho_1^*\omega_1 = \omega_0$, as desired.

3.2.2 Topologically stable Poisson structures

Let Σ be a compact connected oriented 2-dimensional surface. Since there are no non-trivial 3-vectors, any bivector field gives rise to a Poisson structure. Thus, Poisson structures on Σ form a vector space which we denote by $\Pi(\Sigma)$.

For $n \geq 0$ let $\mathscr{G}_n(\Sigma) \subset \Pi(\Sigma)$ be the set of Poisson structures π on Σ such that

- the zero set $Z(\pi) \doteq \{p \in \Sigma | \, \pi(p) = 0\}$ of $\pi \in \mathscr{G}_n(\Sigma)$ consists of n smooth disjoint curves $\gamma_1(\pi), \cdots, \gamma_n(\pi)$;
- π vanishes linearly on each of the curves $\gamma_1(\pi), \cdots, \gamma_n(\pi)$;

In particular, $\mathscr{G}_0(\Sigma)$ is the set of symplectic structures on Σ. Let $\mathscr{G}(\Sigma) \doteq \bigsqcup_{n\geq 0}\mathscr{G}_n(\Sigma)$. The symplectic leaves of a Poisson structure $\pi \in \mathscr{G}(\Sigma)$ are the points in $Z(\pi) = \bigsqcup_{i=1}^n \gamma_i$ (the 0-dimensional leaves) and the connected components of $\Sigma \setminus Z(\pi)$ (the 2-dimensional leaves).

Unless indicated otherwise, throughout the section we denote by ω_0 a symplectic form compatible with the orientation of Σ and by π_0 the corresponding Poisson bivector. Since any π can be written as $\pi = f \cdot \pi_0$ for a function $f \in C^\infty(\Sigma)$, we have $\Pi(\Sigma) = C^\infty(\Sigma) \cdot \pi_0$. The subspace $\mathscr{G}_n(\Sigma)$ corresponds in this way to the product $\mathscr{F}_n(\Sigma) \cdot \pi_0$, where $\mathscr{F}_n(\Sigma)$ is the space of smooth functions for which 0 is a regular value and whose zero set consists of n smooth disjoint curves.

Since $\mathscr{F}(\Sigma) = \bigsqcup_{n\geq 0}\mathscr{F}_n(\Sigma)$ is the set of smooth functions intersecting $0 \in \mathbb{R}$ transversally (see, e.g., [GG73, Def. 4.1]), according to the Elementary Transversality Theorem (see, e.g., Corollary 4.12 in [GG73]), $\mathscr{F}(\Sigma)$ is an open dense subset of $C^\infty(\Sigma)$ in the Whitney C^∞ topology. Therefore, we have the following

Proposition 3.2.4. *The set of Poisson structures $\mathscr{G}(\Sigma)$ is generic inside of $\Pi(\Sigma)$, i.e. $\mathscr{G}(\Sigma)$ is an open dense subset of the space $\Pi(\Sigma)$ of all Poisson structures on Σ endowed with the Whitney C^∞ topology.*

Of course, for some sets of disjoint curves on a surface there are no functions (and, therefore, Poisson structures), vanishing linearly on that set and not zero elsewhere. For example, such is the case of one non-separating curve on a 2-torus.

We will use the following definition

Definition 3.2.5. Two Poisson structures π_1 and π_2 on an oriented manifold P are *globally equivalent* if there is an orientation-preserving Poisson isomorphism $\varphi : (P,\pi_1) \to (P,\pi_2)$.

The main goal of this section is to classify the set $\mathscr{G}(\Sigma)$ of topologically stable structures ion a compact oriented surface Σ up to global equivalence.

First, we will need the following

Lemma 3.2.6. *A topologically stable Poisson structure* $\pi \in \mathscr{G}_n(\Sigma)$ *defines an orientation on each of its zero curves* $\gamma_i \in Z(\pi)$, $i = 1,\dots,n$. *Moreover, this induced orientation on the zero curves of* π *does not depend on the choice of orientation of* Σ.

Proof. Let ω_0 be a symplectic form on Σ, and π_0 be the corresponding Poisson bivector. Since $\pi = f \cdot \pi_0$ and f vanishes linearly on each of $\gamma_i \in Z(\pi)$ and nowhere else, f has a constant sign on each of the 2-dimensional symplectic leaves of π. In particular, f has the opposite signs on two leaves having a common bounding curve γ_i. This defines an orientation on γ_i in the following way. For a non-vanishing vector field X tangent to the curve γ_i, we say that X is *positive* if $\omega_0(X,Y) \geq 0$ for all vector fields Y such that $L_Y f \geq 0$. We say that X is *negative* if $-X$ is positive.

Suppose that X is a vector field tangent to γ_i and positive on γ_i with respect to the chosen orientation of Σ. If ω_0' is a symplectic form inducing the opposite orientation on Σ, then $\omega_0' = -\alpha \cdot \omega_0$ with $\alpha \in C^\infty(\Sigma)$, $\alpha > 0$, and $\pi = -\alpha \cdot f \cdot \pi_0$. Since for Y' such that $L_{Y'}(-\alpha \cdot f) \geq 0$ we have $L_{Y'} f \leq 0$, it follows that $\omega_0(X,Y') < 0$ and, therefore,

$$\omega_0'(X,Y') = -\alpha \cdot \omega_0(X,Y') \geq 0.$$

Hence, if X is positive on γ_i with respect to a chosen orientation of Σ, it is also positive on γ_i with respect to the reverse orientation of Σ. □

We will refer to this orientation of $\gamma_i \in Z(\pi)$ as the *orientation defined by* π.

3.2.3 Diffeomorphism equivalence of sets of disjoint oriented curves

We will use the following definition

Definition 3.2.7. Two sets of smooth disjoint oriented curves $(\gamma_1,\dots,\gamma_n)$ and $(\gamma'_1,\dots,\gamma'_n)$ on a compact oriented surface Σ are called *diffeomorphism equivalent* (denoted by $(\gamma_1,\dots,\gamma_n)\sim(\gamma_1,\dots,\gamma'_n)$) if there is an orientation-preserving diffeomorphism $\varphi:\Sigma\to\Sigma$ mapping the first set onto the second one and preserving the orientations of curves. That is to say, for each $i\in 1,\dots,n$ there exists $j\in 1,\dots,n$ such that $\varphi(\gamma_i)=\gamma'_j$ (as oriented curves).

Let $\mathscr{C}_n(\Sigma)$ be the space of n disjoint oriented curves on Σ and $\mathfrak{M}_n(\Sigma)$ be the moduli space of n disjoint oriented curves on Σ modulo the diffeomorphism equivalence relation, $\mathfrak{M}_n(\Sigma)=\mathscr{C}_n(\Sigma)/\sim$. For a set of disjoint oriented curves $(\gamma_1,\cdots,\gamma_n)$, let $[(\gamma_1,\cdots,\gamma_n)]\in\mathfrak{M}_n(\Sigma)$ denote its class in the moduli space $\mathfrak{M}_n(\Sigma)$. If $\bigsqcup_{i=1}^n\gamma_i=Z(\pi)$ for a Poisson structure π, we will also write $[Z(\pi)]\in\mathfrak{M}_n$ to denote the class of the set of curves $(\gamma_1,\dots,\gamma_n)$ taken with the orientations defined by π.

The topology of the inclusion $Z(\pi)\subset\Sigma$ and the orientations of the zero curves of a topologically stable Poisson structure $\pi\in\mathscr{G}(\Sigma)$ are invariant under orientation-preserving Poisson isomorphisms. In other words, if $\pi,\pi'\in\mathscr{G}_n(\Sigma)$ are globally equivalent, $[Z(\pi)]=[Z(\pi')]\in\mathfrak{M}_n(\Sigma)$.

3.2.4 The modular period invariant

Let $\pi\in\mathscr{G}_n(\Sigma)$ be a topologically stable Poisson structure on a surface Σ. A symplectic form ω_0 compatible with the orientation of Σ is also a volume form on Σ. Let X^{ω_0} be the modular vector field of π with respect to ω_0. Since the flow of X^{ω_0} preserves π, it follows that the restriction of X^{ω_0} to a curve $\gamma_i\in Z(\pi)$ is tangent to γ_i for each $i\in 1,\dots,n$. Since for a different choice ω'_0 of volume form the difference $X^{\omega_0}-X^{\omega'_0}$ is a hamiltonian vector field and, therefore, vanishes on the zero set of π, it follows that the restrictions of X^{ω_0} to $\gamma_1,\cdots,\gamma_n$ are independent of the choice of volume form ω_0. It is apparent from the definition of the modular vector field that it is unchanged if the orientation of the surface is reversed.

Suppose that $\pi\in\Pi(\Sigma)$ vanishes linearly on a curve γ. On a small neighborhood of γ, let θ be the coordinate along the flow of the modular vector field X^{ω_0} with respect to ω_0 such that $X^{\omega_0}=\partial_\theta$. Since π vanishes linearly on γ, there exists an annular coordinate neighborhood (U,z,θ)

of the curve γ such that

$$U = \{(z,\theta) \,|\, |z| < R, \theta \in [0, 2\pi]\}, \tag{3.2.1}$$

$$\gamma = \{(z,\theta) \,|\, z = 0\}, \tag{3.2.2}$$

$$\omega_0|_U = dz \wedge d\theta, \tag{3.2.3}$$

$$\pi|_U = cz\partial_z \wedge \partial_\theta, \quad c > 0. \tag{3.2.4}$$

Using this coordinates, it is easy to verify the following

Claim 3.2.8. The restriction of a modular vector field to the zero curve $\gamma \in Z(\pi)$ (on which the Poisson structure vanishes linearly) is positive with respect to the orientation on γ defined by π (see Claim 3.2.6 for the definition of this orientation).

Definition 3.2.9. (see also [Roy]) For a Poisson structure $\pi \in \Pi(\Sigma)$ vanishing linearly on a curve $\gamma \in Z(\pi)$ define the *modular period of* π *around* γ to be

$$T_\gamma(\pi) \doteq \text{period of } X^{\omega_0}|_\gamma,$$

where X^{ω_0} is the modular vector field of π with respect to a volume form ω_0. Since $X^{\omega_0}|_\gamma$ is independent of the choice of ω_0, the modular period is well-defined.

Using the coordinate neighborhood (U,z,θ) of the curve γ, we obtain

$$T_\gamma(\pi) = \frac{2\pi}{c}, \tag{3.2.5}$$

where $c > 0$ is as in (3.2.4).

It turns out that the modular period of the Poisson structure (3.2.4) on an annulus U is the only invariant under Poisson isomorphisms:

Lemma 3.2.10. *Let* $U(R) = \{(z,\theta) \,|\, |z| < R, \theta \in [0,2\pi]\}$ *and* $U'(R') = \{(z',\theta') \,|\, |z'| < R', \theta' \in [0,2\pi]\}$ *be open annuli with the orientations induced by the symplectic forms* $\omega_0 = dz \wedge d\theta$ *and* $\omega_0' = dz' \wedge d\theta'$ *respectively. Let* $\pi = cz\partial_z \wedge \partial_\theta, c > 0$ *and* $\pi' = c'z'\partial_{z'} \wedge \partial_{\theta'}, c' > 0$ *be Poisson structures on* $U(R)$ *and* $U'(R')$ *for which the modular periods around the zero curves* $\gamma = \{(z,\theta)|z = 0\}$ *and* $\gamma' = \{(z',\theta') \,|\, z' = 0\}$ *are equal,* $T_\gamma(\pi) = T_{\gamma'}(\pi')$*. Then there is an orientation-preserving Poisson isomorphism* $\Phi : (U(R),\pi) \to (U'(R'),\pi')$*.*

Proof. Since the modular periods are equal, $c = c'$. The map $\Phi : (U(R),\pi) \to (U'(R'),\pi')$ given by

$$\Phi(z,\theta) = \left(\frac{R'}{R}z, \theta\right)$$

is a Poisson isomorphism since $\frac{R'}{R}z \cdot \frac{R}{R'}\partial_z \wedge \partial_\theta = z\partial_z \wedge \partial_\theta$. It is easy to see that Φ preserves the orientation. $\square$

The fact that this Poisson isomorphism allows us to change the radius of an annulus will be used later in the proof of the classification theorem.

3.2.5 The regularized Liouville volume invariant

To classify the topologically stable Poisson structures $\mathscr{G}(\Sigma)$ up to orientation-preserving Poisson isomorphisms, we need to introduce one more invariant.

Let $\omega = (\tilde{\pi}^{-1} \otimes \tilde{\pi}^{-1})(\pi)$ be the symplectic form on $\Sigma \setminus Z(\pi)$ corresponding to (the restriction of) $\pi \in \mathscr{G}_n(\Sigma)$. The symplectic volume of each of the 2-dimensional symplectic leaves is infinite because the form ω blows up on the curves $\gamma_1, \cdots, \gamma_n \in Z(\pi)$. However, there is a way to associate a certain finite volume invariant to a Poisson structure in $\mathscr{G}(\Sigma)$, given by the principal value of the integral

$$V(\pi) = P.V. \int_\Sigma \omega.$$

More precisely, let $h \in C^\infty(\Sigma)$ be a function vanishing linearly on $\gamma_1, \cdots, \gamma_n$ and not zero elsewhere. Let $\mathscr{L}$ be the set of 2-dimensional symplectic leaves of π. For $L \in \mathscr{L}$ the boundary ∂L is a union of curves $\gamma_{i_1}, \ldots, \gamma_{i_k} \in Z(\pi)$. (Note that a leaf L can not approach the same curve from both sides). The function h has constant sign on each of the leaves $L \in \mathscr{L}$. For $L \in \mathscr{L}$ and $\varepsilon > 0$ sufficiently small, let

$$L^\varepsilon(h) \doteq L \cap h^{-1}((-\infty, -\varepsilon) \cup (\varepsilon, \infty)), \qquad \partial^\varepsilon L(h) \doteq L \cap h^{-1}(-\varepsilon, \varepsilon)$$

Define

$$V_h^\varepsilon(\pi) \doteq \int_{|h|>\varepsilon} \omega = \sum_{L \in \mathscr{L}} \int_{L^\varepsilon(h)} \omega.$$

Theorem 3.2.11. *The limit $V(\pi) \doteq \lim_{\varepsilon \to 0} V_h^\varepsilon(\pi)$ exists and is independent of the choice of f h.*

Proof. For $i = 1, \ldots, n$, let $U_i = \{(z_i, \theta_i) \,|\, |z_i| < R_i,\ \theta_i \in [0, 2\pi]\}$ be annular coordinate neighborhoods of curves γ_i such that the restriction of π on U_i is given by $\pi_{|U_i} = c_i z_i \partial_{z_i} \wedge \partial_{\theta_i}$, $c_i > 0$ and $U_i \cap Z(\pi) = \gamma_i$. Let $\mathscr{U} = \bigsqcup_{i=1}^n U_i$.

Let h and $\tilde{h}$ be functions vanishing linearly on the curves $\gamma_1, \cdots, \gamma_n$ and not zero elsewhere. On U_i, let $H_{\theta_i}(z_i) \doteq h(z_i, \theta_i)$, $\tilde{H}_{\theta_i}(z_i) \doteq \tilde{h}(z_i, \theta_i)$. Shrink the neighborhoods U_i (if necessary) so that the maps $(z_i, \theta_i) \mapsto (H_{\theta_i}(z_i), \theta_i)$ and $(z_i, \theta_i) \mapsto (\tilde{H}_{\theta_i}(z_i), \theta_i)$ are invertible. Let $\varepsilon > 0$ be sufficiently small so that $\partial^\varepsilon L(h), \partial^\varepsilon L(\tilde{h}) \in \mathscr{U}$. On U_i define

$$g_\varepsilon(\theta_i) \doteq H_{\theta_i}^{-1}(\varepsilon), \quad g_{-\varepsilon}(\theta_i) \doteq H_{\theta_i}^{-1}(-\varepsilon),$$
$$\tilde{g}_\varepsilon(\theta_i) \doteq \tilde{H}_{\theta_i}^{-1}(\varepsilon), \quad \tilde{g}_{-\varepsilon}(\theta_i) \doteq \tilde{H}_{\theta_i}^{-1}(-\varepsilon),$$

so that $g^i_{\pm\varepsilon} \doteq \{g_\varepsilon(\theta_i) \,|\, \theta_i \in [0,2\pi]\} = U_i \cap h^{-1}(\pm\varepsilon)$, $\tilde{g}^i_{\pm\varepsilon} \doteq \{\tilde{g}_{\pm\varepsilon}(\theta_i) \,|\, \theta_i \in [0,2\pi]\} = U_i \cap \tilde{h}^{-1}(\pm\varepsilon)$ are smooth curves in the neighborhood of γ_i. Then the volume $V_h^\varepsilon(\pi)$ can be represented as a sum of the integral of ω over $\Sigma \setminus \mathscr{U}$ (which is independent of ε) and of integrals of ω over some open sets inside of (U_i, z_i, θ_i) $i = 1, \dots, n$:

$$\begin{aligned} V_h^\varepsilon(\pi) &= \int_{|h|>\varepsilon} \omega = \sum_{i=1}^n \int_0^{2\pi} \left(\left(\int_{-R_i}^{g_{-\varepsilon}(\theta_i)} + \int_{g_\varepsilon(\theta_i)}^{R_i} \right) \frac{dz_i}{c_i z_i} \right) d\theta_i + \int_{\Sigma \setminus \mathscr{U}} \omega \\ &= \sum_{i=1}^n \frac{1}{c_i} \int_0^{2\pi} \ln \left| \frac{g_{-\varepsilon}(\theta_i)}{g_\varepsilon(\theta_i)} \right| d\theta_i + \int_{\Sigma \setminus \mathscr{U}} \omega; \end{aligned}$$

$$V_{\tilde{h}}^\varepsilon(\pi) - V_h^\varepsilon(\pi) = \int_{|\tilde{h}|>\varepsilon} \omega - \int_{|h|>\varepsilon} \omega = \sum_{i=1}^n \frac{1}{c_i} \int_0^{2\pi} \ln \left| \frac{\tilde{g}_{-\varepsilon}(\theta_i)}{g_{-\varepsilon}(\theta_i)} \cdot \frac{g_\varepsilon(\theta_i)}{\tilde{g}_\varepsilon(\theta_i)} \right| d\theta_i;$$

Since H_{θ_i} and $\tilde{H}_{\theta_i}$ are smooth invertible functions, the limits

$$\lim_{\varepsilon \to 0} \left| \frac{g_{-\varepsilon}(\theta_i)}{g_\varepsilon(\theta_i)} \right|, \quad \lim_{\varepsilon \to 0} \left| \frac{\tilde{g}_{-\varepsilon}(\theta_i)}{g_{-\varepsilon}(\theta_i)} \cdot \frac{g_\varepsilon(\theta_i)}{\tilde{g}_\varepsilon(\theta_i)} \right|$$

exist and equal to 1. Thus, $V(\pi) = \lim_{\varepsilon \to 0} V_h^\varepsilon(\pi)$ exists and is independent of the choice of h. $\square$

Remark 3.2.12. The fact that the principle value of the integral $\int_\Sigma \omega$ is well-defined is a consequence of the following more general statement (having essentially the same proof). Let M be a compact manifold, and let Ω be a volume form on M. Let furthermore f be a function, which has zero as a regular value. Then one can define the principal value of the integral $\int_M \frac{\Omega}{f}$ in a way that is independent of the choice of coordinates. This seems to be well-known to specialists, but we could not locate a precise reference.

Hence $V(\pi) \in \mathbb{R}$ is a global equivalence invariant of a Poisson structure $\pi \in \mathscr{G}(\Sigma)$ on a compact oriented surface which we call the *regularized Liouville volume* since in the case of a symplectic structure (i.e., $\pi \in \mathscr{G}_0(\Sigma)$) it is exactly the Liouville volume. If we reverse the orientation of Σ, the regularized volume invariant changes sign.

3.2.6 The classification theorem

Theorem 3.2.13. *Topologically stable Poisson structures $\mathscr{G}_n(\Sigma)$ on a compact connected oriented surface Σ are completely classified (up to an orientation-preserving Poisson isomorphism) by the following data:*

1. *The equivalence class* $[Z(\pi)] \in \mathfrak{M}_n(\Sigma)$ *of the set* $Z(\pi) = \bigsqcup_{i=1}^n \gamma_i$ *of zero curves with orientations defined by* π*;*

2. *The modular periods around the zero curves* $\{\gamma_i \mapsto T_{\gamma_i}(\pi) |\ i = 1, \ldots, n\}$*;*

3. *The regularized Liouville volume* $V(\pi)$*;*

In other words, two Poisson structures $\pi, \pi' \in \mathscr{G}_n(\Sigma)$ *are globally equivalent if and only if their sets of oriented zero curves are diffeomorphism equivalent, the modular periods around the corresponding curves are the same, and the regularized Liouville volumes are equal.*

To prove this result we will need the following

Lemma 3.2.14. *Let D be a connected* 2*-dimensional manifold, and* ω_1, ω_2 *be two symplectic forms on D inducing the same orientation and such that*

- $\omega_1|_{D\setminus K} = \omega_2|_{D\setminus K}$ *for a compact set* $K \subset D$*;*
- $\int_D \omega_1 - \omega_2 = 0$*;*

Then there exists a symplectomorphism $\varphi : (D, \omega_1) \to (D, \omega_2)$ *such that* $\varphi|_{D\setminus K} = id$*.*

Proof. (Moser's trick). Let $\omega_t \doteq \omega_1 \cdot (1-t) + \omega_2 \cdot t$ for $t \in [0,1]$. Since $\omega_1|_{D\setminus K} = \omega_2|_{D\setminus K}$, the form $\Delta(\omega) \doteq \omega_2 - \omega_1$ is compactly supported ($\mathrm{supp}(\Delta(\omega)) \subset K$). Since $\int_D \omega_1 - \omega_2 = 0$, the class of $\Delta(\omega)$ in the second de Rham cohomology with compact support $H^2_{\text{deRham,compact}}(K^o)$ is trivial (here K^o denotes the interior of K). Hence $\Delta(\omega) = d\mu$ for a 1-form $\mu \in \Omega^1_{\text{compact}}(K^o)$. Then for $v_t \doteq -\tilde{\omega}_t^{-1}(\mu)$ we have

$$L_{v_t}\omega_t = d\iota_{v_t}\omega_t + \iota_{v_t}d\omega_t = d\iota_{v_t}\omega_t = -\Delta(\omega).$$

Therefore,

$$L_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0 \tag{3.2.6}$$

Let ρ_t be the flow of the time-dependent vector field v_t. Since

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(L_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) = 0,$$

$\rho_t^*\omega_t = \omega_1$ for all $t \in [0,1]$. Since $v_t = 0$ outside of K, it follows that $\rho_t|_{D\setminus K} = \mathrm{id}$. Define $\varphi = \rho_1$. Then $\varphi_{|D\setminus K} = \mathrm{id}$ and $\varphi^*\omega_2 = \rho_1^*\omega_2 = \omega_1$ as desired. ☐

We now have all the ingredients for the proof of the classification Theorem 3.2.13.

Proof. (of Theorem 3.2.13) The class $[(\gamma_1,\cdots,\gamma_n)]\in\mathfrak{M}_n(\Sigma)$, the modular periods $\{\gamma_i\mapsto T_{\gamma_i}(\pi)\,|\,i=1,\ldots,n\}$ and the total volume $V(\pi)$ are clearly global invariants of a Poisson structure in $\mathscr{G}_n(\Sigma)$.

Suppose that for Poisson structures $\pi,\pi'\in\mathscr{G}_n(\Sigma)$ we have $[Z(\pi)]=[Z(\pi')]\in\mathfrak{M}_n(\Sigma)$. Since this implies that there exists an orientation-preserving diffeomorphism $\varphi\in\mathrm{Diff}(\Sigma)$ such that $\varphi(Z(\pi))=Z(\pi')$ (where $Z(\pi)$ is considered as the set of oriented curves), we may from now on assume $Z(\pi)=Z(\pi')=\bigsqcup_{i=1}^n\gamma_i$.

For each $i=1,\ldots,n$, let $U_i(R_i)\doteq\{(z_i,\theta_i)\,|\,|z_i|<R_i\}$ and $U_i'(R_i')\doteq\{(z_i',\theta_i')\,|\,|z_i'|<R_i'\}$ be annular neighborhoods of the curve γ_i, such that

$$\pi|_{U_i}=c_iz_i\partial_{z_i}\wedge\partial_{\theta_i},\quad \pi'|_{U_i'}=c_iz_i'\partial_{z_i'}\wedge\partial_{\theta_i'}.$$

The radii R_i and R_i' should be small enough so that $U_i(R_i)\cap Z(\pi)=\gamma_i$, $U_i'(R_i')\cap Z(\pi)=\gamma_i'$.

Let $\mathscr{U}=\bigsqcup_{i=1}^nU_i(R_i)$, $\mathscr{U}'=\bigsqcup_{i=1}^nU_i'(R_i')$. Since $V(\pi)=V(\pi')$, we can choose the radii R_i,R_i' of the neighborhoods $U_i(R_i),U_i'(R_i')$ in such a way that for each 2-dimensional leaf $L\in\mathscr{L}$ the following non-compact symplectic manifolds

$$D(L)\doteq L\setminus\overline{\mathscr{U}}\qquad\text{and}\qquad D'(L)\doteq L\setminus\overline{\mathscr{U}'}$$

have equal (finite) symplectic volumes: $\int_{D(L)}\omega=\int_{D'(L)}\omega'$, where ω (respectively, ω') is the symplectic form on $\Sigma\setminus Z(\pi)$ corresponding to the Poisson structure π (respectively, π'.)

Consider the coverings of (Σ,π) and (Σ,π') by the sets $\{(D(L),U_i(R_i))\,|\,i=1,\ldots,n;L\in\mathscr{L}\}$ and $\{(D'(L),U_i'(R_i')\,|\,i=1,\ldots,n;L\in\mathscr{L}\}$ respectively. Since $T_{\gamma_i}(\pi)=T_{\gamma_i}(\pi')$ and the orientations of γ_i defined by π and π' coincide, by Lemma 3.2.10, there exist orientation-preserving Poisson isomorphisms $\psi_i:U_i(R_i)\to U_i'(R_i')$ of these neighborhoods given in local coordinates by $\psi_i(r_i,\theta_i)=\left(\frac{R_i'}{R_i}r_i,\theta_i\right)$. Choosing a small $\varepsilon_i<R_i$ and $\varepsilon_i'=\varepsilon_i\cdot\frac{R_i}{R_i'}$, we obtain Poisson isomorphisms $\psi_i:U_i(R_i)\to U_i'(R_i')$ mapping $U_i(R_i)\setminus U_i(R_i-\varepsilon_i)$ onto $U_i'(R_i')\setminus U_i'(R_i'-\varepsilon_i')$.

The Poisson isomorphisms $\psi_1,\ldots,\psi_n$ map $L\cap\mathscr{U}$ onto $L\cap\mathscr{U}'$. Therefore, we can extend $\psi_1,\cdots,\psi_n$ to a diffeomorphism Ψ of the surface Σ (first extend it as a diffeomorphism of class C^1 and then smooth it out to a C^∞-diffeomorphism; see [Mun63] for details on smoothing maps) such that Ψ preserves the oriented zero curves and the 2-dimensional leaves. (Without the assumption that π and π' define the same orientation on γ_i, $i=1,\ldots,n$ it might happen that (for $\gamma_i\in\partial L$) the image of $L\cap U_i$ under ψ_i does not belong to L. As a result, it might not be possible to extend $\psi_1,\ldots,\psi_n$ to a diffeomorphism $\Psi\in\mathrm{Diff}(\Sigma)$, as in Example 3.2.16 below).

Let $\widetilde{\mathscr{U}}=\bigsqcup_{i=1}^nU_i(R_i-\varepsilon_i)$, $\widetilde{\mathscr{U}'}=\bigsqcup_{i=1}^nU_i'(R_i'-\varepsilon_i')$. For $L\in\mathscr{L}$, consider the non-compact connected manifold $\tilde{D}\equiv\widetilde{D(L)}\doteq L\setminus\overline{\widetilde{\mathscr{U}}}$ with symplectic structures $\omega_1=\omega|_{\tilde{D}}$ and $\omega_2=\Psi^*\omega'|_{\tilde{D}}$.

Since by construction $\omega_1|_{\tilde{D}\setminus K} = \omega_2|_{\tilde{D}\setminus K}$ for $K \doteq \widetilde{D(L)} \cap \overline{\mathscr{U}}$ and $\int_{\tilde{D}} \omega_1 = \int_{\tilde{D}} \omega_2$, the result now follows by application of Lemma 3.2.14 to $\widetilde{D(L)}$ for each $L \in \mathscr{L}$. $\square$

Remark 3.2.15. Given $\pi, \pi' \in \mathscr{G}_n(\Sigma)$, one can ask if (Σ, π) and (Σ, π') are equivalent by an arbitrary (possibly orientation-reversing) Poisson isomorphism. Fix an orientation-reversing diffeomorphism $\nu : \Sigma \to \Sigma$. Then π and π' are equivalent by an orientation-reversing diffeomorphism if and only if $\nu_*\pi$ and π' are equivalent by an orientation-preserving diffeomorphism. It is not hard to see that $T_{\gamma_i}(\pi) = T_{\nu(\gamma_i)}(\nu_*\pi)$ for all $\gamma_i \in Z(\pi)$ and $V(\pi) = -V(\nu_*\pi)$. Thus the question of equivalence by orientation reversing maps can be reduced to the orientation-preserving context of Theorem 3.2.13.

Example 3.2.16. Let ω and $\omega' = -\omega$ be two symplectic structures on a compact oriented surface. Then ω and ω' are Poisson isomorphic by an orientation-reversing diffeomorphism, but not by an orientation-preserving diffeomorphism.

There are, of course, similar examples of structures with non-trivial sets of linear degeneracy. Consider the unit 2-sphere S^2 with the cylindrical polar coordinates (z, θ) away from its poles. Let $\omega_0 = dz \wedge d\theta$ be a symplectic form on S^2 with the corresponding Poisson bivector π_0. Let $\pi, \pi' \in \mathscr{G}_2(S^2)$ be the Poisson structures given by

$$\pi = (z-a)(z-b)\partial_z \wedge \partial_\theta, \quad -1 < b < a < 1$$

and $\pi' = -\pi$. Choose a and b in such a way that $V(\pi) = V(\pi') = 0$. Let $\gamma_1 = \{(z,\theta) | z = a\}$ and $\gamma_2 = \{(z,\theta) | z = b\}$ be the zero curves of π, π'. On both γ_1 and γ_2 the orientations defined by π and π' are opposite to each other. Let $L_{\text{top}} = \{(z,\theta) | a < z < 1\}$, $L_{\text{middle}} = \{(z,\theta) | b < z < a\}$ and $L_{\text{bottom}} = \{(z,\theta) | -1 < z < b\}$ be the 2-dimensional leaves (common for both structures). The structures π and π' can not be Poisson isomorphic in an orientation-preserving way since such a diffeomorphism would have to exchange the two-dimensional disks L_{top} and L_{bottom} with the annulus L_{middle}. On the other hand, (S^2, π) and (S^2, π') are clearly Poisson isomorphic by an orientation-reversing diffeomorphism $(z,\theta) \mapsto (z, -\theta)$.

3.2.7 Poisson cohomology of topologically stable Poisson structures

In this section we compute the Poisson cohomology of a given topologically stable Poisson structure on a compact connected oriented surface and describe its relation to the infinitesimal deformations and the classifying invariants introduced above. (For generalities on Poisson cohomology see, e.g., [Vai94]).

First, recall the following

Lemma 3.2.17. *(e.g., Roytenberg [Roy]) The Poisson cohomology of an annular neighborhood* $U = \{(z,\theta) \,|\, |z| < R,\, \theta \in [0,2\pi]\}$ *of the curve* γ *on which* $\pi_{|U} = z\partial_z \wedge \partial_\theta$ *is given by*

$$H^0_\pi(U,\pi_{|U}) = span\langle 1\rangle = \mathbb{R}$$
$$H^1_\pi(U,\pi_{|U}) = span\langle \partial_\theta, z\partial_z\rangle = \mathbb{R}^2$$
$$H^2_\pi(U,\pi_{|U}) = span\langle z\partial_z \wedge \partial_\theta\rangle = \mathbb{R}$$

Thus, $H^0_\pi(U)$ is generated by constant functions. The first cohomology $H^1_\pi(U)$ is generated by the modular class (∂_θ is the modular vector field of $\pi_{|U}$ with respect to $\omega_0 = dz\wedge d\theta$) and the image of the first de Rham cohomology class of U (spanned by $d\theta$) under the homomorphism $\tilde{\pi} : H^1_{\text{deRham}}(U) \to H^1_\pi(U)$, which is injective in this case. The second cohomology is generated by $\pi_{|U}$ itself.

Let $\pi \in \mathscr{G}_n(\Sigma)$ be a topologically stable Poisson structure on Σ. Since a Casimir function on Σ must be constant on all connected components of $\Sigma \setminus Z(\pi)$, by continuity it must be constant everywhere. Hence $H^0_\pi(\Sigma) = \mathbb{R} = \text{span}\langle 1\rangle$.

We will (inductively) use the Mayer-Vietoris sequence of Poisson cohomology (see, e.g., [Vai94]) to compute $H^1_\pi(\Sigma)$ and $H^2_\pi(\Sigma)$.

Let U_i be an annular neighborhood of the curve $\gamma_i \in Z(\pi)$ such that $U_i \cap Z(\pi) = \gamma_i$. Let $V_0 \doteq \Sigma$ and define inductively $V_i \doteq V_{i-1} \setminus \gamma_i$ for $i = 1,\dots,n$. Consider the cover of V_{i-1} by open sets U_i and V_i. To compute the first cohomology, we consider the first two rows of the Mayer-Vietoris exact sequence of Poisson cohomology associated to this cover:

$$0 \to H^0_\pi(V_{i-1}) \xrightarrow{\alpha^0_i} H^0_\pi(U_i) \oplus H^0_\pi(V_i) \xrightarrow{\beta^0_i} H^0_\pi(U_i \cap V_i) \xrightarrow{\delta^0_i}$$
$$\to H^1_\pi(V_{i-1}) \xrightarrow{\alpha^1_i} H^1_\pi(U_i) \oplus H^1_\pi(V_i) \xrightarrow{\beta^1_i} H^1_\pi(U_i \cap V_i) \xrightarrow{\delta^1_i} \dots$$

By exactness, $H^1_\pi(V_{i-1}) \simeq \delta^0_i(H^0_\pi(U_i\cap V_i)) \oplus \ker\beta^1_i$, where

$$\beta^1_i\left([\chi]_{U_i}, [\nu]_{V_i}\right) = [\chi - \nu]_{U_i\cap V_i} \qquad \chi \in \mathfrak{X}^1_\pi(U_i), \nu \in \mathfrak{X}^1_\pi(V_i), d_\pi\chi = 0, d_\pi\nu = 0,$$

and $[X]_W$ denotes the class of the (Poisson) vector field $X|_W$ in $H^1_\pi(W)$, for $W = U_i, V_i$.

By Lemma 3.2.17, $H^1_\pi(U_i) \simeq \tilde{\pi}(H^1_{\text{deRham}}(U_i)) \oplus \text{span}\langle\partial_{\theta_i}\rangle \simeq \mathbb{R}^2$. Since $U_i \cap V_i$ is a union of two symplectic annuli, $H^1_\pi(U_i\cap V_i) = \tilde{\pi}(H^1_{\text{deRham}}(U_i\cap V_i)) = \mathbb{R}^2$. Therefore,

$$H^1_\pi(V_{i-1}) \simeq \delta^0_i(H^0_\pi(U_i\cap V_i)) \oplus \text{span}\langle\partial_{\theta_i}\rangle \oplus \ker\left(\beta^1_i|_{\tilde{\pi}(H^1_{\text{deRham}}(U_i))\oplus H^1_\pi(V_i)}\right). \qquad (3.2.7)$$

Consider also the long exact sequence in de Rham cohomology associated to the same cover

$$\begin{aligned} 0 &\to H^0_{\text{deRham}}(V_{i-1}) \xrightarrow{a_i^0} H^0_{\text{deRham}}(U_i) \oplus H^0_{\text{deRham}}(V_i) \xrightarrow{b_i^0} H^0_{\text{deRham}}(U_i \cap V_i) \xrightarrow{d_i^0} \\ &\to H^1_{\text{deRham}}(V_{i-1}) \xrightarrow{a_i^1} H^1_{\text{deRham}}(U_i) \oplus H^1_{\text{deRham}}(V_i) \xrightarrow{b_i^1} H^1_{\text{deRham}}(U_i \cap V_i) \xrightarrow{d_i^1} \dots \end{aligned}$$

By exactness, we have $H^1_{\text{deRham}}(V_{i-1}) \simeq d_i^0(H^0_{\text{deRham}}(U_i \cap V_i)) \oplus \ker b_i^1$. Since $V_n = \Sigma \setminus Z(\pi)$ is symplectic, $H^*_\pi(V_n) = \tilde{\pi}(H^*_{\text{deRham}}(V_n))$. This together with $H^0_\pi(U_n) \simeq H^0_{\text{deRham}}(U_n)$, $H^0_\pi(U_n \cap V_n) = \tilde{\pi}(H^0_{\text{deRham}}(U_n \cap V_n))$ implies $\text{Im}(\delta^0_n) = \tilde{\pi}(\text{Im}(d^0_n))$ and, therefore,

$$\ker\left(\beta^1_{n-1}|_{\tilde{\pi}(H^1_{\text{deRham}}(U_n) \oplus H^1_\pi(V_n))}\right) = \tilde{\pi}(\ker(b^1_{n-1})).$$

Hence, from (3.2.7) it follows

$$H^1_\pi(V_{n-1}) \simeq \tilde{\pi}(\text{Im}(d_i)) \oplus \text{span}\langle \partial_{\theta_n} \rangle \oplus \tilde{\pi}(H^1_{\text{deRham}}(V_{n-1})).$$

For $i = n-2$, we have

$$\begin{aligned} H^1_\pi(V_{n-2}) \simeq \text{span}\langle \partial_{\theta_{n-1}} \rangle \oplus \ker\left(\beta^1_{n-1}|_{\tilde{\pi}(H^1_{\text{deRham}}(U_{n-1}) \oplus \left(\tilde{\pi}(H^1_{\text{deRham}}(V_{n-1})) \oplus \tilde{\pi}(\text{Im}(d_i^0))\right))}\right) = \\ = \text{span}\langle \partial_{\theta_{n-1}} \rangle \oplus \text{span}\langle \partial_{\theta_n} \rangle \oplus \tilde{\pi}(H^1_{\text{deRham}}(V_{n-2})). \end{aligned}$$

Working inductively (from $i = n-1$ to $i = 0$), we obtain

$$H^1_\pi(\Sigma) \simeq \mathbb{R}^{n+2g} = \bigoplus_{i=1}^{n} \text{span}\langle \partial_{\theta_i} \rangle \oplus H^1_{\text{deRham}}(\Sigma),$$

where g is the genus of the surface Σ.

To compute the second Poisson cohomology, it is more convenient to consider the covering of Σ by $V \doteq V_n$ and $U \doteq \bigsqcup_{i=1}^n U_i$. The second and third rows of the associated Mayer-Vietoris exact sequence are given by

$$\begin{aligned} &\to H^1_\pi(\Sigma) \xrightarrow{\alpha^1} H^1_\pi(U) \oplus H^1_\pi(V) \xrightarrow{\beta^1} H^1_\pi(U \cap V) \xrightarrow{\delta^1} \\ &\to H^2_\pi(\Sigma) \xrightarrow{\alpha^2} H^2_\pi(U) \oplus H^2_\pi(V) \xrightarrow{\beta^2} H^2_\pi(U \cap V) \to 0. \end{aligned}$$

Since $H^2_\pi(U \cap V) = 0$ and $H^2_\pi(V)$, it follows that

$$H^2_\pi(\Sigma) \simeq \text{Im}(\delta^1) \oplus) H^2_\pi(U) = \delta_1\left(\frac{H^1_\pi(U \cap V)}{\beta^1(H^1_\pi(U) \oplus H^1_\pi(V))}\right).$$

We have $H^1_\pi(U \cap V) \simeq \tilde{\pi}(H^1_{\text{deRham}}(U \cap V))$, $H^1_\pi(U) \simeq \tilde{\pi}(H^1_{\text{deRham}}(U)) \oplus \text{span}\ \langle \partial_{\theta_1}, \dots, \partial_{\theta_n} \rangle$, $H^1_\pi(V) \simeq \tilde{\pi}(H^1_{\text{deRham}}(V))$. Therefore, $\beta^1(H^1_\pi(U) \oplus H^1_\pi(V)) = \beta^1(\tilde{\pi}(H^1_{\text{deRham}}(U)) \oplus \tilde{\pi}(H^1_{\text{deRham}}(V)))$. We obtain

$$H^2_\pi(\Sigma) = \delta\left(\frac{\tilde{\pi}(H^1_{\text{deRham}}(U \cap V))}{\beta^1(\tilde{\pi}(H^1_{\text{deRham}}(U)) \oplus \tilde{\pi}(H^1_{\text{deRham}}(V))}\right). \tag{3.2.8}$$

Consider the Mayer-Vietoris exact sequence for de Rham cohomology associated to the same cover:

$$\begin{aligned}
&\to H^1_{\text{deRham}}(\Sigma) \xrightarrow{a^1} H^1_{\text{deRham}}(U) \oplus H^1_{\text{deRham}}(V) \xrightarrow{b^1} H^1_{\text{deRham}}(U\cap V) \xrightarrow{d^1}\\
&\to H^2_{\text{deRham}}(\Sigma) \xrightarrow{a^2} H^2_{\text{deRham}}(U) \oplus H^2_{\text{deRham}}(V) \to H^2_{\text{deRham}}(U\cap V) \to 0
\end{aligned}$$

This implies

$$H^2_{\text{deRham}}(\Sigma) = d_1\left(\frac{H^1_{\text{deRham}}(U\cap V)}{b^1(H^1_{\text{deRham}}(U)\oplus H^1_{\text{deRham}}(V))}\right). \tag{3.2.9}$$

Comparing (3.2.8) and (3.2.9), we obtain

$$H^2_\pi(\Sigma) \simeq \mathbb{R}^{n+1} = H^2_\pi(U) \oplus \tilde{\pi}(H^2_{\text{deRham}}(\Sigma)) = \mathbb{R}^{n+1}.$$

where the first n generators are the Poisson structures of the form

$$\pi_i \doteq \pi\cdot\{\text{bump function around the curve } \gamma_i\},\quad i=1,\dots,n \tag{3.2.10}$$

and the last generator is the standard non-degenerate Poisson structure π_0 on Σ. Therefore, we have proved the following

Theorem 3.2.18. *Let $\pi\in\mathscr{G}_n(\Sigma)$ be a topologically stable Poisson structure on a compact connected oriented surface Σ of genus g. The Poisson cohomology of π is given by*

$$\begin{aligned}
H^0_\pi(\Sigma,\pi) &= span\langle 1\rangle = \mathbb{R}\\
H^1_\pi(\Sigma,\pi) &= span\langle X^{\omega_0}(\pi_1),\cdots,X^{\omega_0}(\pi_n)\rangle \oplus \tilde{\pi}(H^1_{deRham}(\Sigma)) = \mathbb{R}^{n+2g}\\
H^2_\pi(\Sigma,\pi) &= span\langle \pi_1,\cdots,\pi_n\rangle \oplus \tilde{\pi}(H^2_{deRham}(\Sigma)) = span\langle \pi_0;\pi_1,\dots,\pi_n\rangle = R^{n+1},
\end{aligned}$$

where π_0 is a non-degenerate Poisson structure on Σ , π_i, $i=1,\dots,n$ is a Poisson structure vanishing linearly on $\gamma_i\in Z(\pi)$ and identically zero outside of a neighborhood of γ_i; $X^{\omega_0}(\pi_i)$ is the modular vector field of π_i with respect to the standard symplectic form ω_0 on Σ.

Notice that the dimensions of the cohomology spaces depend only on the number of the zero curves and not on their positions. In particular, the Poisson cohomology as a vector space does not depend on the homology classes of the zero curves of the structure. Recall (see, e.g., [Vai94]) that the Poisson cohomology space $H^*_\pi(P)$ has the structure of an associative graded commutative algebra induced by the operation of wedge multiplication of multivector fields. A direct computation verifies the following

Proposition 3.2.19. *The wedge product on the cohomology space $H^*_\pi(\Sigma,\pi)$ of a topologically stable Poisson structure on Σ is determined by*

$$
\begin{aligned}
&[1]\wedge[\chi]=[\chi], \quad [\chi]\in H^*_\pi(\Sigma,\pi)\\
&[X^{\omega_0}(\pi_i)]\wedge[X^{\omega_0}(\pi_j)]=0, \quad i,j=1,\dots,n\\
&[X^{\omega_0}(\pi_i)]\wedge[\tilde{\pi}(\alpha)]=[\alpha(X^{\omega_0}(\pi_i))\cdot\pi_i]=\left(\frac{1}{T_{\gamma_i}(\pi)}\int_{\gamma_i}\alpha\right)\cdot[\pi_i]\\
&[\tilde{\pi}(\alpha)]\wedge[\tilde{\pi}(\alpha')]=\tilde{\pi}(\overline{\alpha}\wedge\overline{\alpha'}),\\
&[\chi]\wedge H^2_\pi(\Sigma,\pi)=0, \quad [\chi]\in H^1_\pi(\Sigma)\oplus H^2_\pi(\Sigma),
\end{aligned}
$$

where $\alpha,\alpha'\in\Omega^1(\Sigma),\ d\alpha=d\alpha'=0.$

(Here bar denotes the class of its argument in the de Rham cohomology and the brackets [ ] denote the class in the Poisson cohomology). We should mention that the wedge product $\wedge$ in de Rham cohomology is dual to the intersection product in homology [BT82].

This computation allows one to compute the number of zero curves γ_k, which determine non-zero homology classes. To see this, we note that $[\pi_k]\in H^1_\pi(\Sigma)\wedge H^1_\pi(\Sigma)$ iff there exists a 1-form α such that $\int_{\gamma_k}\alpha\neq 0$, i.e., γ_k is non-zero in homology. If Σ is not a sphere, $H^1_{\mathrm{deRham}}(\Sigma)$ is non-zero. Since the intersection form on $H^1_{\mathrm{deRham}}(\Sigma)$ is non-degenerate (implementing Poincare duality), it follows that $H^1_{\mathrm{deRham}}(\Sigma)\wedge H^1_{\mathrm{deRham}}(\Sigma)\neq 0$. Thus in the case that Σ is not a sphere, $H^1_\pi(\Sigma)\wedge H^1_\pi(\Sigma)$ has the set

$$\{\pi_0\}\cup\{\pi_k:\gamma_k \text{ is nontrivial in homology}\}$$

as a basis and so the number of curves γ_k, which are non-trivial in homology, is just $\dim(H^1_\pi(\Sigma)\wedge H^1_\pi(\Sigma))-1$. In the case that Σ is a sphere, all γ_k are of course topologically trivial.

Proposition 3.2.20. *The Schouten bracket on the cohomology space $H^*_\pi(\Sigma,\pi)$ of a topologically stable Poisson structure on Σ is determined by*

$$
\begin{aligned}
&[H^0_\pi(\Sigma),H^*_\pi(\Sigma)]=0\\
&[H^1_\pi(\Sigma),H^1_\pi(\Sigma)]=0\\
&[[\tilde{\pi}(d\varphi(\gamma_i))],[\pi_0]]=-[\pi_0], \quad i,j=1,\dots,n\\
&[[\tilde{\pi}(d\varphi(\gamma_i))],[\pi_j]]=0, \quad i,j=1,\dots,n\\
&[[X^{\omega_0}(\pi_i)],H^2_\pi(\Sigma)]=0, \quad i=1,\dots,n\\
&[H^2_\pi(\Sigma),H^2_\pi(\Sigma)]=0.
\end{aligned}
$$

The generators of $H^2_\pi(\Sigma,\pi)$ can be interpreted as infinitesimal deformations of the Poisson structure π which change the classifying invariants.

Corollary 3.2.21. *Let $\pi \in \mathscr{G}_n(\Sigma)$, $Z(\pi) = \bigsqcup_{i=1}^n \gamma_i$. The following $n+1$ one-parameter families of infinitesimal deformations form a basis of $H^2_\pi(\Sigma,\pi)$*

(1) $\pi \mapsto \pi + \varepsilon \cdot \pi_0$;

(2) $\pi \mapsto \pi + \varepsilon \cdot \pi_i$, $i = 1,\dots,n$;

Each of these deformations changes exactly one of the classifying invariants of the Poisson structure: $\pi \mapsto \pi + \varepsilon \cdot \pi_0$ changes the regularized Liouville volume and $\pi \mapsto \pi + \varepsilon \cdot \pi_i$ changes the modular period around the curve γ_i for each $i = 1,\dots,n$.

3.2.8 Example: topologically stable Poisson structures on the sphere

It would be interesting to describe the moduli space of the space of topologically stable Poisson structures on a compact oriented surface up to orientation-preserving diffeomorphisms. The first step would be the description of the moduli space $\mathfrak{M}_n$ of n disjoint oriented curves on Σ. However, this problem is already quite difficult for a general surface. Here we will in detail consider the simplest example of topologically stable Poisson structures on the sphere.

Let $(\gamma_1,\cdots,\gamma_n)$ be a set of disjoint curves on S^2. Let $\mathscr{S}_1,\cdots,\mathscr{S}_{n+1}$ be the connected components of $S^2 \setminus (\gamma_1,\cdots,\gamma_n)$. To the configuration of curves $(\gamma_1,\cdots,\gamma_n)$ we associate a graph $\Gamma(\gamma_1,\cdots,\gamma_n)$ in the following way. The vertices $v_1,\dots,v_n$ of the graph correspond to the connected components $\mathscr{S}_1,\dots,\mathscr{S}_n$. Two vertices v_i and v_j are connected by an edge e_k iff γ_k is the common bounding curve of the regions $\mathscr{S}_i$ and $\mathscr{S}_j$.

Claim 3.2.22. For a set of disjoint curves $\gamma_1,\dots,\gamma_n$ on S^2 the graph $\Gamma(\gamma_1,\dots,\gamma_n)$ is a tree.

Proof. Let $e_i \in E(\Gamma(\gamma_1,\dots,\gamma_n))$ be an edge of the graph corresponding to the curve γ_i. Since $S^2 \setminus \gamma_i$ is a union of two open sets, it follows that $\Gamma(\gamma_1,\dots,\gamma_n) \setminus e_i$ (i.e., the graph $\Gamma(\gamma_1,\dots,\gamma_n)$ with the edge e_i removed) is a union of two disjoint graphs. Since this is true for any e_i, $i = 1,\dots,n$, the graph is a tree. $\square$

Choose an orientation on S^2 and a symplectic form ω_0 (with the Poisson bivector π_0) which induces this orientation. Let $\pi \in \mathscr{G}_n(\Sigma)$ be a topologically stable Poisson structure. The function $f = \pi/\pi_0$ has constant signs on the 2-dimensional symplectic leaves. Let $\Gamma(\gamma_1,\dots,\gamma_n)$ be the tree associated to the zero curves $\gamma_1,\dots,\gamma_n$ of π as described above. Assign to each vertex v_i a sign (plus or minus) equal to the sign of the function f on the corresponding symplectic leaf $\mathscr{S}_i$ of

π. The properties of π imply that for any edge of this tree its ends are assigned the opposite signs. We will call the tree associated to the zero curves of π with signs associated to its vertices the *signed tree* $\Gamma(\pi)$ of the Poisson structure π.

Consider the map $T_e(\pi) : E(\Gamma(\pi)) \to \mathbb{R}^+$ which for each edge $e \in E(\Gamma(\pi))$ gives a period $T_e(\pi)$ of a modular vector field of π around the zero curve corresponding to this edge. The classification Theorem 3.2.13 implies

Theorem 3.2.23. *The topologically stable Poisson structures $\pi \in \mathcal{G}_n(S^2)$ on the sphere are completely classified (up to an orientation-preserving Poisson isomorphism) by the signed tree $\Gamma(\pi)$, the map $e \mapsto T_e(\pi)$, $e \in E(\Gamma(\pi))$ and the regularized Liouville volume $V(\pi)$. In other words, $\pi_1, \pi_2 \in \mathcal{G}_n(S^2)$ are globally equivalent if and only if the corresponding $\Gamma(\pi_i)$, $\{e \mapsto T_e(\pi_i)\}$, $V(\pi_i)$ are the same (up to automorphisms of signed trees with positive numbers attached to their edges).*

The moduli space of topologically stable Poisson structures in $\mathcal{G}_n(S^2)$ up to Poisson isomorphisms is

$$\mathcal{G}_n(S^2)/(\text{Poisson isomorphisms}) \simeq \left(\bigsqcup_{\Gamma_{n+1}} (\mathbb{R}^+)^n / \mathrm{Aut}(\Gamma, T_e) \right) \times \mathbb{R},$$

where $\mathrm{Aut}(\Gamma, T_e)$ is the automorphism group of the signed tree Γ with $n+1$ vertices and with positive numbers T_e attached to its edges $e \in E(\Gamma(\pi))$. The moduli space has dimension $n+1$ and is coordinatized by $\{T_e(\pi) | \, e \in E(\Gamma(\pi))\}$ and $V(\pi)$.

A particular case of topologically stable Poisson structures on S^2, the $SU(2)$-covariant structures vanishing on a circle on S^2, were considered by D.Roytenberg in [Roy]. In cylindrical coordinates (z, θ) on the unit sphere these structures are given by

$$\pi_c = a(z-c)\partial_z \wedge \partial_\theta \qquad \text{for } |c| < 1, a > 0$$

The modular period around the zero curve (a "horizontal" circle $\gamma = \{(z, \theta) | \, z = c\}$) and the regularized Liouville volume are given by

$$T_{\gamma=\{(z,\theta)|\,z=c\}}(\pi) = \frac{2\pi}{a},$$
$$V(\pi) = \frac{2\pi}{a} \ln \frac{1+c}{1-c}.$$

Note that for a non-degenerate Poisson structure $\pi_c = a(z-c)\partial_z \wedge \partial_\theta$, $|c| > 1$ the total Liouville volume is given by the same formula, $V(\pi) = \frac{2\pi}{a} \ln \left| \frac{1+c}{1-c} \right|$.

Corollary 3.2.24. *Let $T \in \mathbb{R}^+$ and $V \in \mathbb{R}$. A Poisson structure $\pi \in \mathscr{G}_1(S^2)$ with the modular period T and the regularized total volume V is globally equivalent to the Poisson structure which in coordinates (z,θ) on S^2 is given by*

$$\pi(T,V) = \frac{2\pi}{T}\left(z - \frac{e^{V/T}-1}{e^{V/T}+1}\right)\partial_z \wedge \partial_\theta$$

and vanishes linearly on the circle $z = \frac{e^{V/T}-1}{e^{V/T}+1}$.

Utilizing Poisson cohomology, D. Roytenberg [Roy, Corollary 4.3.3, 4.3.4] has previously obtained that the structures π_c, $-1 < c < 1$ are non-trivial infinitesimal deformations of each other. Similarly, he proved that for each c, π_c admits no infinitesimal rescalings. Using Theorem 3.2.13, we get the following improvement of his results:

Corollary 3.2.25. *(a) The Poisson structures $\pi_c = a(z-c)\partial_z \wedge \partial_\theta$ and $\pi_{c'} = a(z-c')\partial_z \wedge \partial_\theta$ are globally equivalent iff $c = c'$.*
(b) For $\alpha \in \mathbb{R} \setminus \{0\}$, the Poisson structures π_c and $\alpha\pi_c$ are equivalent via an orientation-preserving Poisson isomorphism (respectively, arbitrary Poisson isomorphism) if and only if $\alpha = 1$ (respectively, $|\alpha| = 1$). In particular, π_c admits no rescalings.

3.3 Toward a classification of Poisson structures with higher order singularities.

If a Poisson structure vanishes non-linearly on its zero set, a finite number of invariants may not be enough to achieve even a local classification. For example, the second Poisson cohomology of the structure $\{x,y\} = x^n$ with $n \geq 2$ on $\mathbb{R}^2$ is infinite-dimensional. Therefore, a finite number of invariants would not be enough to distinguish all of its infinitesimal deformations. As we shall see below, the situation is much nicer if the higher-order singularities are isolated. For example, in the case of isolated quadratic singularities, we will exhibit a finite number of classifying invariants.

3.3.1 Structures with isolated higher order singularities.

For $\alpha = 1,2$, let π_α be a Poisson structure on a compact oriented surface Σ_α, and let $Z(\pi_\alpha) \doteq \{p \in \Sigma_\alpha | \pi_\alpha(p) = 0\}$ be its zero set. Assume that π_α vanishes linearly on $Z(\pi_\alpha)$ except at a finite number of special points $p_\alpha^j \in Z(\pi_\alpha)$, $j = 1,\dots,m_\alpha$, where the degeneracies are of higher order. Let now U_α^j be neighborhoods of the special points p_α^j, so that each U_α^j is diffeomorphic to a disc, and $U_\alpha^j \cap U_\alpha^i = \emptyset$ for $i \neq j$. Let $\Sigma'_\alpha \doteq \Sigma_\alpha \setminus (U_\alpha^1 \cup \cdots \cup U_\alpha^{m_\alpha})$. Then Σ'_α is an open surface, and $\pi'_\alpha \doteq \pi_\alpha|_{\Sigma'_\alpha}$ has only linear degeneracies. Let furthermore $\{V_\alpha^j\}_{j=1}^{m_\alpha}$ be disjoint open neighborhoods of $\{U_\alpha^j\}_{j=1}^{m_\alpha}$, and let $W_\alpha^j \doteq V_\alpha^j \cap \Sigma'_\alpha$, for $\alpha = 1,2,\ j = 1,\ldots,m_\alpha$. Also, denote by $\gamma_\alpha^1,\ldots,\gamma_\alpha^{n_\alpha}$ the connected components of $Z(\pi'_\alpha)$, and by $T_{\gamma_\alpha^j}(\pi'_\alpha)$ the *modular flow time* of π_α along γ_α^j (which for an open curve is defined to be the maximal flow time of a modular vector field along this curve, similarly to Definition 3.2.9).

3.3.2 The regularized Liouville volume.

Assume that the flow time $T_{\gamma_\alpha^j}$ along each connected component γ_{α^j} of $Z(\pi'_\alpha)$ is finite. One can assign to π'_α a regularized volume invariant $V(\pi'_\alpha : \Sigma' \subset \Sigma)$, in much the same way that was done for a Poisson structure with linear degeneracies on a closed surface. However, we must keep track of the behavior of π'_α at the "infinities" of the open surface $\Sigma' \subset \Sigma$. We therefore emphasize the possible dependence of this invariant on the embedding of Σ' into Σ in our notation.

To define $V(\pi'_\alpha : \Sigma' \subset \Sigma)$, note that because the Poisson structure degenerates linearly on each γ_α^j, and because the flow time along each γ_α^j is finite, we can find a finite covering of Σ'_α by open sets X_i, $i \in I$, Y_j, $j \in J$, Z_k, $k \in K$, so that:

1. For all $i \in I$, X_i are pre-compact and π'_α is non-zero on the closure of X_i;

2. For all $j \in J$, $(Y_j, \pi'_\alpha|_{Y_j}) \cong (S^1 \times (-r_j, r_j), r\partial_r \wedge \partial_\theta)$ (here θ is the periodic coordinate on S^1 and r is a coordinate on the open interval $(-r_j, r_j)$);

3. For all $k \in K$, $(Z_k, \pi'_\alpha|_{Z_k}) \cong ((-l_k, l_k) \times (-c_k, c_k), y\partial_x \wedge \partial_y)$, where x is the coordinate on $(-l_k, l_k)$ and y is the coordinate on $(-c_k, c_k)$.

We note that the coordinates θ and x are canonical (since the modular vector fields of the associated Poisson structures are ∂_θ and ∂_x, respectively). Now, because π'_α is nonzero on the closure of X_i, its restriction to X_i comes from a finite-volume symplectic structure; thus the volume $V(X_i, \pi'_\alpha)$ is well-defined. To define the volume of π'_α on Y_i, let ω_α be the symplectic form corresponding to π'_α on $\Sigma' \setminus Z(\pi_{\alpha'})$ and set

$$V(Y_i, \pi'_\alpha) = \lim_{\varepsilon \to 0} \int_{Y_i \cap \{p : |h(p)| > \varepsilon\}} \omega_\alpha$$

for any function h so that $h\omega_\alpha$ extends smoothly to a non-zero symplectic form in a neighborhood of $Z(\pi'_\alpha)$. It was shown in the course of the proof of Theorem 3.2.11 that this limit is independent of the choice of h.

The idea of the proof for the case of $V(Z_k, \pi'_\alpha)$ is similar. Let

$$V(Z_k, \pi'_\alpha) = \lim_{\varepsilon \to 0} \int_{Z_k \cap \{p: |h(p)| > \varepsilon\}} \omega_\alpha,$$

where h is such that $h\omega_\alpha$ extends to a smooth nonzero form on a neighborhood of Σ' in Σ. Let $\tilde{h}$ be another choice of such h, and set $H_x(y) \doteq h(x,y)$, $\tilde{H}_x(y) \doteq \tilde{h}(x,y)$, where (x,y) are coordinates as in the definition of Z_k. We may assume that the maps $(x,y) \mapsto (x, H_x(y))$ and $(x,y) \mapsto (x, \tilde{H}_x(y))$ are invertible. Let $\varepsilon > 0$ be sufficiently small. Define

$$g_\varepsilon(x) \doteq H_x^{-1}(\varepsilon), \quad g_{-\varepsilon}(x) \doteq H_x^{-1}(-\varepsilon),$$
$$\tilde{g}_\varepsilon(x) \doteq \tilde{H}_x^{-1}(\varepsilon), \quad \tilde{g}_{-\varepsilon}(x) \doteq \tilde{H}_x^{-1}(-\varepsilon),$$

so that $g^i_{\pm\varepsilon} \doteq \{g_\varepsilon(x) | x \in (-l_k, l_k)\} = Z_k \cap h^{-1}(\pm\varepsilon)$, $\tilde{g}^i_{\pm\varepsilon} \doteq \{\tilde{g}_{\pm\varepsilon}(x) | x \in (-l_k, l_k)\} = U_i \cap \tilde{h}^{-1}(\pm\varepsilon)$ are smooth curves in the neighborhood of γ_i. Then

$$\begin{aligned} V_h^\varepsilon &= \int_{Z_k \cap \{p: |h(p)| > \varepsilon\}} \omega_\alpha = \int_{-l_k}^{l_k} \left(\left(\int_{-c_k}^{g_{-\varepsilon}(x)} + \int_{g_\varepsilon(x)}^{c_k} \right) \frac{dy}{y} \right) dx \\ &= \int_{-l_k}^{l_k} \ln \left| \frac{g_{-\varepsilon}(x)}{g_\varepsilon(x)} \right| dx; \end{aligned}$$

$$V_{\tilde{h}}^\varepsilon - V_h^\varepsilon = \int_{-l_k}^{l_k} \ln \left| \frac{\tilde{g}_{-\varepsilon}(x)}{g_{-\varepsilon}(x)} \cdot \frac{g_\varepsilon(x)}{\tilde{g}_\varepsilon(x)} \right| dx;$$

Since H_x and $\tilde{H}_x$ are smooth invertible functions on the *closed* interval $[-l_k, l_k]$, the limits

$$\lim_{\varepsilon \to 0} \left| \frac{g_{-\varepsilon}(x)}{g_\varepsilon(x)} \right|, \quad \lim_{\varepsilon \to 0} \left| \frac{\tilde{g}_{-\varepsilon}(x)}{g_{-\varepsilon}(x)} \cdot \frac{g_\varepsilon(x)}{\tilde{g}_\varepsilon(x)} \right|$$

exist and equal to 1. Thus, $V(Z_k, \pi'_\alpha) = \lim_{\varepsilon \to 0} V_h^\varepsilon(\pi)$ exists and is independent of the choice of h.

We can therefore set

$$V(\pi'_\alpha : \Sigma' \subset \Sigma) \doteq \sum_i V(X_i, \pi'_\alpha) + \sum_j V(Y_j, \pi'_\alpha) + \sum_k V(Z_k, \pi'_\alpha).$$

It is easily seen that this sum does not depend on the choice of the covering of Σ'.

3.3.3 Global classification.

Suppose that for $\alpha = 1, 2$ the number of special points of π_α is m_α, and the number of connected components of $Z(\pi'_\alpha)$ is n_α. We record the following lemma, whose proof proceeds exactly as in the case of Theorem 3.2.13.

Lemma 3.3.1. *With the above notation, assume that $\phi : \Sigma_1 \to \Sigma_2$ is a diffeomorphism, so that:*

1. ϕ *maps a neighborhood* W_1^j *onto* W_2^j *and* $\phi_*\pi_1'|_{W_1^j} = \pi_2'|_{W_2^j}$ *for all* $j = 1,\dots,m$;

2. ϕ *maps* γ_1^i *to* γ_2^i *for all* $i = 1,\dots,n$;

3. *the modular flow times along corresponding curves are the same,* $T_{\gamma_1^i}(\pi_1') = T_{\gamma_2^i}(\pi_2') < \infty$ *for all* $i = 1,\dots,n$;

4. *the regularized volume invariants of* π_1' *and* π_2' *are the same,* $V(\pi_1') = V(\pi_2')$;

Then there exists a diffeomorphism $\theta : \Sigma_1' \to \Sigma_2'$, *such that* $\theta|_{W_1^j} = \phi|_{W_1^j}$ *for all* $j = 1,\dots,m$ *and* $\theta_*\pi_1' = \pi_2'$. *Moreover,* θ *can be extended to a diffeomorphism of* Σ_1 *with* Σ_2.

Proof. (Sketch; see also the proof of Theorem 3.2.13) Because of the assumptions (1)–(3), we may extend ϕ (using the flows of modular vector fields to transport the neighborhoods W_1^j around the curves γ_1^i) to a neighborhood W_1 containing W_1^j for all $j = 1,\dots,m$ and γ_1^i for all $i = 1,\dots,n$ in such a way that $\phi_*\pi_1'|_{W_1} = \pi_2'|_{W_2}$, where $W_2 = \phi(W_1)$. Now we proceed exactly as in the proof of Theorem 3.2.13 to extend ϕ to the complement of W_1, using condition (4). Since $\theta = \phi$ was not modified on each W_1^j, it can be extended to a diffeomorphism of Σ_1 onto Σ_2 (e.g., by defining it to be ϕ on each V_1^j). $\square$

The utility of this Lemma is explained by the following Corollary:

Corollary 3.3.2. *With the above notation, assume that there exists a diffeomorphism* $\phi : \Sigma_1 \to \Sigma_2$, *so that*

1. ϕ *maps* U_1^j *onto* U_2^j *and* $\phi_*\pi_1|_{U_1^j} = \pi_2|_{U_2^j}$ *for all* $j = 1,\dots,m$;

2. ϕ *maps* γ_1^i *onto* γ_2^i *for all* $i = 1,\dots,n$;

3. *the modular flow times along corresponding curves are the same,* $T_{\gamma_1^i}(\pi_1') = T_{\gamma_2^i}(\pi_2') < \infty$;

4. *the regularized volume invariants of* π_1' *and* π_2' *are the same,* $V(\pi_1' : \Sigma' \subset \Sigma_1) = V(\pi_2' : \phi(\Sigma') \subset \Sigma_2)$;

Then there exists a diffeomorphism $\theta : \Sigma_1 \to \Sigma_2$ *so that* $\theta|_{U_1^j} = \phi|_{U_1^j}$ *for all* j *and* $\theta_*\pi_1 = \pi_2$.

3.3.4 Localization.

The Corollary 3.3.2 allows one to reduce the question of whether π_1 and π_2 are Poisson-isomorphic to local considerations. To do so, one proceeds as follows:

- *Step I*: find neighborhoods U_1^j and U_2^j, $j = 1, \dots, m$ of special points so that
  - (3) and (4) of Corollary 3.3.2 are satisfied;
  - there exists a diffeomorphism $\phi : \Sigma_1 \to \Sigma_2$ satisfying (2) and mapping U_1^j to U_2^j;
  - there exist neighborhoods $O_1^j \subset U_1^j$ of the special point p_1^j such that the maps $\phi|_{U_j^1 \setminus O_j}$ are Poisson diffeomorphisms (thus perhaps failing (1) on O_1^j);
- *Step II*: for each j, check whether $\phi : U_1^j \to U_2^j$ can be perturbed to a Poisson isomorphism, keeping it the same on $U_1^j \setminus O_1^j$ (a local question);

If Steps I and Steps II can be carried out for a given pair π_1 and π_2 of Poisson structures with isolated higher order singularities, Corollary 3.3.2 implies that π_1 and π_2 are Poisson-isomorphic.

Proposition 3.3.3. *To find the diffeomorphism in Step I, it is necessary and sufficient that there exists a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$ carrying the zero set $Z(\pi_1)$ onto the zero set $Z(\pi_2)$ and mapping the higher-order degeneracy points p_1^j of π_1 to the respective points p_2^j of π_2, $j = 1, \dots, m$.*

Proof. Indeed, given ψ, we can first choose neighborhoods O_1^j of p_1^j and O_2^j of $\psi(p_1^j) = p_2^j$ in such a way that the modular lengths of the components of $Z(\pi_1) \setminus (O_1^1 \cup \cdots \cup O_1^m)$ are the same as the corresponding components of $Z(\pi_2) \setminus (O_2^1 \cup \cdots \cup O_2^m)$. By further modifying O_1^j (e.g., by removing a small disk near its boundary and away from $Z(\pi_1)$) we may assume that the regularized Liouville volume invariants of $\pi_1|_{\Sigma_1 \setminus (O_1^1 \cup \cdots \cup O_1^m)}$ and $\pi_2|_{\Sigma_2 \setminus (O_2^1 \cup \cdots \cup O_2^m)}$ are the same. We may furthermore assume, by modifying ψ in a neighborhood U_1^j of O_1^j, that it is a Poisson diffeomorphism from $U_1^j \setminus O_1^j$ onto $U_2^j \setminus O_2^j = \psi(U_1^j) \setminus \psi(O_1^j)$. Since the modular length of a segment of a curve and the regularized Liouville volume are invariant under Poisson maps, it follows that the modular length of $\gamma_1^j \cap (U_1^j \setminus O_1^j)$ is the same as that of $\gamma_2^j \cap (U_2^j \setminus O_2^j)$, and the volume of $U_1^j \setminus O_1^j$ is the same as that of $U_2^j \setminus O_2^j$. Thus conditions (2), (3) and (4) are fulfilled for this choice of U_α^j, $\alpha = 1, 2$ and ψ. Thus we have indeed carried out Step 1. □

3.3.5 Local classification in the quadratic case.

Let $\pi = f\pi_0$ be a Poisson structure on a surface Σ which vanishes linearly on its zero set $Z(\pi) \subset \Sigma$ except possibly at a finite number of points $p^1, \dots, p^m \in Z(\pi)$, where it could have quadratic zeros; i.e., $f(x,y) = \lambda^j(x^2 + \sigma^j y^2)$ for some $\lambda^j \neq 0$ and $\sigma^j \in \{\pm 1\}$ in a local coordinate system (x,y) around p^j. We call p^j, $j = 1, \dots, m$ the *special points* of the structure π. If $\sigma^j = 1$ (resp., $\sigma^j = -1$), the special point p^j is called *elliptic* (resp., *hyperbolic*). Clearly, the special points (as a subset of $Z(\pi)$) is an invariant of π.

Our aim is to extend the classification results of Section 3.2 to such structures.

By Proposition 3.3.3, the question of classification of such structures can be reduced to local considerations. Let U_1 and U_2 be two open disk neighborhoods around $(0,0) \in \mathbb{R}^2$, and let

$$\pi_\alpha = \lambda_\alpha(x^2 + \sigma_\alpha y^2)\partial_x \wedge \partial_y|_{U_\alpha} \quad \text{for } \alpha = 1,2$$

Our first question is: when can there be a Poisson isomorphism of (U_1, π_1) and (U_2, π_2). It turns out that $(\lambda_\alpha, \sigma_\alpha)$ is a complete local invariant for such a structure. To prove this, we consider separately the cases of elliptic and hyperbolic special points.

The case of $(x^2 + y^2)\partial_x \wedge \partial_y$.

In this case the zero sets of π_1 and π_2 consist of the origin $(0,0)$.

Proposition 3.3.4. *(U_1, π_1) is Poisson-isomorphic to (U_2, π_2) in an orientation-preserving way if and only if $\lambda_1 = \lambda_2$. Moreover, if $\lambda_1 = \lambda_2$, then the Poisson isomorphism can be chosen so as to extend to all of $\mathbb{R}^2$ and be identity outside of any prescribed open set containing U_1 and U_2.*

Proof. For $\alpha = 1,2$, let X_α be the modular vector field of π_α with respect to the volume form $\frac{1}{2}dx \wedge dy$. It is not hard to check that X_α is a linear vector field given by

$$X_\alpha = \lambda_\alpha(x\partial_y - y\partial_x).$$

Its flow at time t is the rotation, $F_{X_\alpha}(t) : (r\cos\theta, r\sin\theta) \mapsto (r\cos(\theta + \lambda_\alpha t), r\sin(\theta + \lambda_\alpha t))$. Since the flow of X_α fixes $(0,0)$, it follows that the flow defines a linear operator T_α on the tangent space to $\mathbb{R}^2$ at the origin given by

$$T_\alpha(v) = \frac{d}{dt}(F_X(t)_*(v)), \quad v \in T_{(0,0)}\mathbb{R}^2.$$

The structure of T_α as a linear operator on the tangent space is an invariant of π_α. It is easily seen that in our coordinates T_α is given by the matrix

$$T_\alpha = \begin{pmatrix} 0 & \lambda_\alpha \\ -\lambda_\alpha & 0 \end{pmatrix}.$$

Thus the eigenvalues of T_α (equal to $\pm i\lambda_\alpha$) are an invariant of π_α, which we will call the *modular eigenvalues invariant.* It follows that if π_1 and π_2 are isomorphic, then $\lambda_1 = \lambda_2$.

Assume now that $\lambda_1 = \lambda_2 = \lambda$ (or, equivalently, that the modular eigenvalues invariant is the same). We must prove that given two neighborhoods U_1 and U_2 of $(0,0)$ diffeomorphic to a disk, there exists a Poisson isomorphism of (U_1,π_1) with (U_2,π_2), which can be extended to all of $\mathbb{R}^2$. Denote by $O(r)$ the open disk $\{(x,y)\in\mathbb{R}^2 : x^2+y^2<r^2\}$. Choose r_1 and r_2 so that $O(r_i)\subset U_i$ and the symplectic volumes of $U_1\setminus O(r_1)$ and $U_2\setminus O(r_2)$ are equal. Then applying Moser's theorem, we can find a Poisson isomorphism θ from $U_1\setminus O(r_1)$ onto $U_2\setminus O(r_2)$, so that θ extends to all of $\mathbb{R}^2$ and on a neighborhood of the boundary $\partial O(r_1)$ maps the point (x,y) to the point $\frac{r_2}{r_1}(x,y)$.

It remains to note that the map

$$(x,y)\mapsto \frac{r_2}{r_1}(x,y), \quad \sqrt{x^2+y^2}<r_1$$

extends θ to a map from all of U_1 onto U_2, having the desired properties. $\square$

We note as a corollary that if a Poisson structure π on a compact oriented surface Σ has at a point $p^1\in\Sigma$ a singularity of the type $\lambda(x^2+y^2)\partial_x\wedge\partial_y$, and some singularities at points $p^2,\dots,p^m\in\Sigma$, then for any choice of neighborhoods $U^j\ni p^j$, $j=2,\dots,m$, and any prescribed number $V\in\mathbb{R}$, we can find a neighborhood U^1 of p^1 so that the regularized Liouville volume of $\Sigma\setminus(U^1\cup\cdots\cup U^m)$ is exactly V.

The case of $(x^2-y^2)\partial_x\wedge\partial_y$.

In this case the situation is more intricate. Suppose that a Poisson structure is given by $\pi = \lambda(x^2-y^2)\partial_x\wedge\partial_y$ on a neighborhood U of $(0,0)\in\mathbb{R}^2$. The zero set of π can be identified with the intersection of U and the lines $\ell_1(\pi)\doteq\{(x,y):x=y\}$ and $\ell_2(\pi)\doteq\{(x,y):x=-y\}$. As we remarked before, the restriction $X|_{Z(\pi)}$ of a modular vector field X depends only on the Poisson structure (and not on the choice of a volume form with respect to which it is calculated). The subsets $\ell_1(\pi)$ and $\ell_2(\pi)$ are canonically distinguished, since along one of them $X|_{Z(\pi)}$ points toward the point of intersection $(0,0)=\ell_1(\pi)\cap\ell_2(\pi)$, while along the other it points away from it. The modular vector field of π with respect to the area form $\frac{1}{2}dx\wedge dy$ is given by

$$X=\lambda(x\partial_y+y\partial_x).$$

Once again, the flow of X fixes $(0,0)$; the associated linear operator on the tangent space at $(0,0)$ is given by the matrix

$$T=\lambda\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and has eigenvalues $\pm\lambda$, which we again call the *modular eigenvalues invariant* and denote by $\pm\lambda(\pi)$ for a Poisson structure π.

Although the total modular flow time between the origin and any other point in $Z(\pi)$ is infinite, for any two points on $\ell_j(\pi)$, $j = 1,2$ which are distinct from the origin one can define a modular flow time between these points. Let $p = (a,(-1)^{j-1}a)$, $q = (b,(-1)^{j-1}b)$ (with $a,b \neq 0$) be two points on the same line $\ell_j(\pi)$. Choose a parameterization $\gamma(t) = (g(t),(-1)^{j-1}g(t))$ of $\ell_j(\pi)$, so that $\gamma(-1) = p$, $\gamma(1) = q$. If p and q are on the same connected component of $\ell_j(\pi)\setminus\{(0,0)\}$, the following integral

$$I(p,q) \doteq \int_{-1}^{1} \frac{\gamma'(t)}{X(\gamma(t))}dt$$

is well-defined and independent of the parameterization $\gamma(t)$, since $X(s) = 0$ only if $s = (0,0)$ (here $\gamma'(t)$ denotes the tangent vector to the parametrized curve, and the ratio of $\gamma'(t)$ by $X(\gamma(t))$ makes sense because these two vectors are parallel).

Assume now that p and q are on the opposite sides of $(0,0) \in \ell_j(\pi)$. Let t_0 be such that $\gamma(t_0) = (0,0)$. Define the *modular flow time* by

$$I(p,q) \doteq \lim_{\varepsilon\to 0}\left(\int_{-1}^{t_0-\varepsilon} \frac{\gamma'(t)}{X(\gamma(t))}dt + \int_{t_0+\varepsilon}^{1} \frac{\gamma'(t)}{X(\gamma(t))}dt\right).$$

Although the length of time it takes to get from p to $(0,0)$ along the flow of the modular vector field is infinite, the corresponding time it takes to flow from $(0,0)$ to q is also infinite, but has an opposite sign (since one goes against the flow of the modular vector field to get to q). This is similar to the situation in physics, when the amount of energy it takes to move from a point in space to a singularity of a potential is infinite, yet it takes a finite energy to "tunnel" across the potential well. For this reason, $I(p,q)$ could be thought of as the "energy" it takes to move from p to q.

We now claim that the definition of $I(p,q)$ is independent of the parameterization $\gamma(t)$ and that the limit in the definition always exists. This actually follows from Remark 3.2.12, but we prefer to give a direct proof. Note first that $X(\gamma(t)) = \lambda g(t)\cdot((-1)^{j-1}\partial_x + \partial_y) = (-1)^{j-1}\lambda g(t)\cdot(\partial_x + (-1)^{j-1}\partial_y)$ and, therefore,

$$\frac{\gamma'(t)}{X(\gamma(t))} = (-1)^{j-1}\frac{g'(t)}{g(t)}.$$

It follows that

$$(-1)^{j-1}\int_{-1}^{t_0-\varepsilon} \frac{g'(t)}{g(t)}dt = \log(|g(t_0-\varepsilon)|) - \log(|g(-1)|),$$

$$(-1)^{j-1}\int_{t_0+\varepsilon}^{1} \frac{g'(t)}{g(t)}dt = \log(|g(1)|) - \log(|g(t_0+\varepsilon)|).$$

Thus

$$I(p,q) = (-1)^{j-1}\lim_{\varepsilon\to 0}\left(\log\left|\frac{g(1)}{g(-1)}\right| - \log\left|\frac{g(t_0+\varepsilon)}{g(t_0-\varepsilon)}\right|\right) = (-1)^{j-1}\log\left|\frac{g(1)}{g(-1)}\right|,$$

since

$$\lim_{\varepsilon\to 0}\left|\frac{g(t_0+\varepsilon)}{g(t_0-\varepsilon)}\right| = \left|\frac{g'(t_0)}{g'(t_0)}\right| = 1$$

by L'Hopital's rule.

Thus, $I(p,q)$ is defined for any pair of points $p,q \in \ell_j$ and depends only on the Poisson structure π and the choice of points p,q. In particular, associated to (U,π) we can consider the numbers

$$R_j(\pi) \doteq \sup_{p,q\in\ell_j} I(p,q) = (-1)^{j+1}\log\left|\frac{\sup\{x : (x,y)\in U,\, x=(-1)^{j+1}y\}}{\inf\{x : (x,y)\in U,\, x=(-1)^{j+1}y\}}\right|, \quad j=1,2$$

The numbers $R_1(\pi)$ and $R_2(\pi)$ are invariants of the Poisson structure π on U. Note that the lines ℓ_j (and thus of the numbers R_j) are canonically distinguished (since the modular vector field along $\ell_1(\pi)$ always points away from $(0,0)$, while on $\ell_2(\pi)$ it always points toward $(0,0)$).

Finally, let $\omega = \frac{1}{x^2-y^2}dx\wedge dy$ be the symplectic form on $U\setminus Z(\pi)$ corresponding to the restriction of π to $U\setminus Z(\pi)$. By the discussion above, the tangent space at $(0,0)$ has two preferred one-dimensional subspaces, spanned by the eigenvectors of T (which are in our case the vectors proportional to $v_\pm = \partial_x \pm \partial_y$). Define the regularized Liouville volume of (U,π) by

$$V(\pi) \doteq P.V.\int_U \omega = \lim_{\varepsilon\to 0}\left(\int_{f^{-1}((\infty,-\varepsilon^2))\cap U}\omega + \int_{f^{-1}((\varepsilon^2,\infty))\cap U}\omega\right), \tag{3.3.1}$$

where f is any function so that

- $f(0,0)=0$, $df|_{(0,0)}=0$, $v_+v_+f = -v_-v_-f$, $v_+v_-f=0$;
- $f\cdot\omega$ extends to a non-degenerate (symplectic) 2-form on U;
- $f\le 0$ on the region above the lines ℓ_1 and ℓ_2 ;

We claim that the limit exists and is independent of the choice of f. To prove this, it is convenient to change the coordinate system as follows. Let $x' \doteq x+y$, $y' \doteq y-x$. Then $\pi = 2x'y'\partial_{x'}\wedge\partial_{y'}$ and $\omega = \frac{1}{2x'y'}dx\wedge dy$. Let

$$V_f^\varepsilon = V_f^\varepsilon(\pi) = \left(\int_{f^{-1}((\infty,-\varepsilon))\cap U}\omega + \int_{f^{-1}((\varepsilon,\infty))\cap U}\omega\right)$$

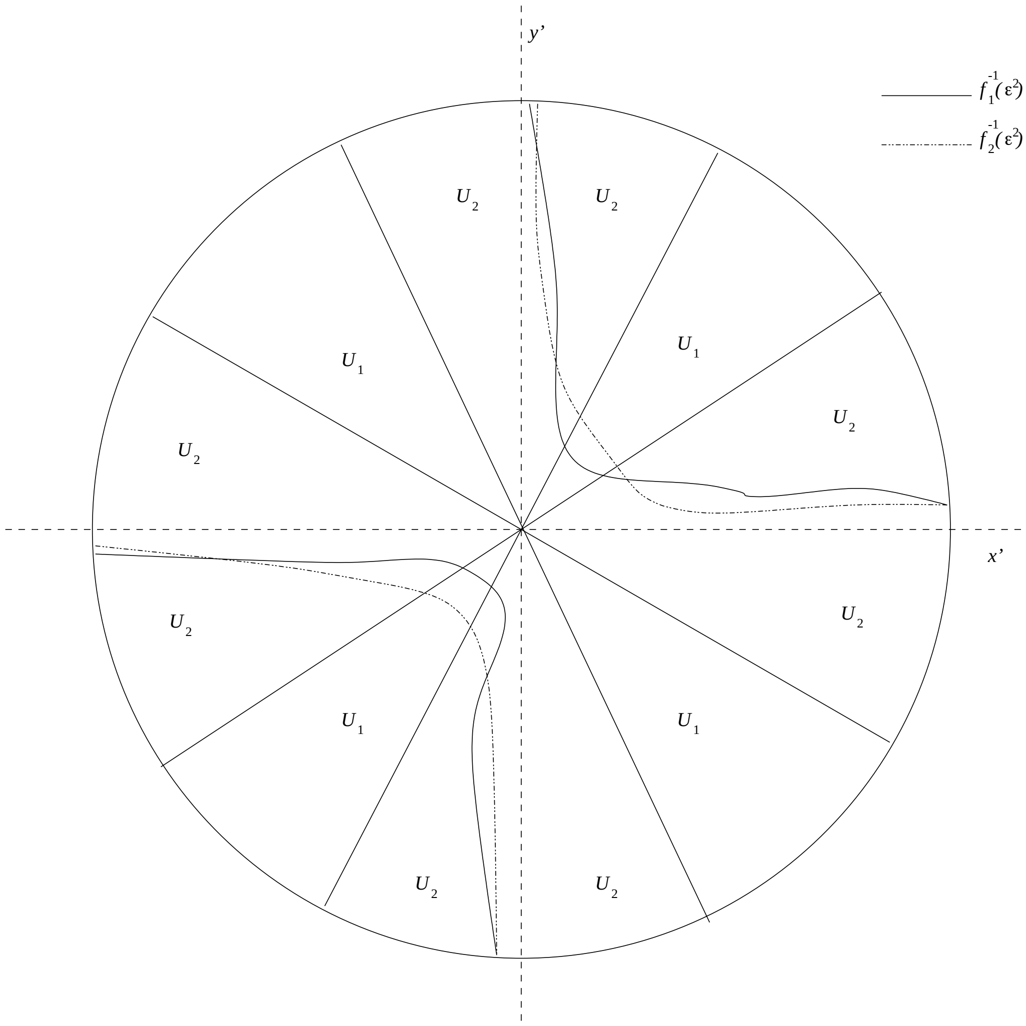

Figure 3.3.1: Definition of regularized volume in the quadratic case.

Let f and $\tilde{f}$ be two functions satisfying the conditions above. We will show that $\lim_{\varepsilon\to 0}(V_f^\varepsilon - V_{\tilde{f}}^\varepsilon) = 0$. The assumptions on f and $\tilde{f}$ imply that

$$f(x',y') = x'y' \cdot g(x',y') \tag{3.3.2}$$

$$\tilde{f}(x',y') = x'y' \cdot \tilde{g}(x',y') \tag{3.3.3}$$

for some smooth functions $g,\tilde{g}$. Let $U_1 = U \cap \left\{(x',y') \in \mathbb{R}^2 : \frac{|x'|}{2} < |y'| < 2|x'|\right\}$ and $U_2 = U \setminus U_1$ (see Figure 3.3.1). Let $\alpha = x'/y'$. Thus $\frac{1}{2} < |\alpha| < 2$ on U_1. We have

$$V_f^\varepsilon - V_{\tilde{f}}^\varepsilon = \int_{R_1\cup R_2} \frac{dx'dy'}{x'y'},$$

where $R_1 = \left\{(x',y') \in U_1 : f(x',y') > \varepsilon^2,\ \tilde{f}(x',y') < \varepsilon^2 \text{ or } f(x',y') < \varepsilon^2,\ \tilde{f}(x',y') > \varepsilon^2\right\}$ and $R_2 = \left\{(x',y') \in U_2 : f(x',y') > \varepsilon^2,\ \tilde{f}(x',y') < \varepsilon^2 \text{ or } f(x',y') < \varepsilon^2,\ \tilde{f}(x',y') > \varepsilon^2\right\}$. Over the region R_1, we switch to the coordinate system (α, y'). Since $dx'dy' = y'd\alpha dy'$, we have

$$\begin{aligned}
\left|\int_{R_1} \frac{dx'dy'}{x'y'}\right| &= \left|\int_{R_1} \frac{d\alpha dy'}{\alpha y'}\right| \\
&\leq \int_{\frac{1}{2}\leq|\alpha|\leq 2} \frac{d\alpha}{|\alpha|}\left|\int_{f_\alpha^{-1}(\varepsilon^2)}^{\tilde{f}_\alpha^{-1}(\varepsilon^2)} \frac{dy}{y}\right| = \int_{\frac{1}{2}\leq|\alpha|\leq 2} \frac{d\alpha}{|\alpha|}\left|\log\left|\frac{\tilde{f}_\alpha^{-1}(\varepsilon^2)}{f_\alpha^{-1}(\varepsilon^2)}\right|\right|,
\end{aligned}$$

where $f_\alpha(y') = f(\alpha y', y')$. Using (3.3.2) and (3.3.3), we find that

$$\begin{aligned}
(f_\alpha^{-1}(\varepsilon^2))^2 &= \frac{\varepsilon^2}{\alpha}(1 + O_\alpha(\varepsilon)) \\
(\tilde{f}_\alpha^{-1}(\varepsilon^2))^2 &= \frac{\varepsilon^2}{\alpha}(1 + \tilde{O}_\alpha(\varepsilon)).
\end{aligned}$$

Thus

$$\log\left|\frac{\tilde{f}_\alpha^{-1}(\varepsilon^2)}{f_\alpha^{-1}(\varepsilon^2)}\right| = O'_\alpha(\varepsilon).$$

Since $\frac{1}{2} < |\alpha| < 2$ on U_1, we find that therefore the integral $\int \frac{d\alpha}{|\alpha|}\left|\log\left|\frac{\tilde{f}_\alpha^{-1}(\varepsilon^2)}{f_\alpha^{-1}(\varepsilon^2)}\right|\right| \to 0$ as $\varepsilon \to 0$, and thus

$$\int_{R_1} \frac{dx'dy'}{x'y'} \to 0.$$

Let ρ be the diameter of U. Write $R_2 = R_{2,x} \cup R_{2,y}$, where $R_{2,x} = \{(x',y') \in R_2 : |x'| > |y'|\}$ and $R_{2,y} = \{(x',y') \in R_2 : |x'| < |y'|\}$. Denote by $r_x(\varepsilon)$ the infimum

$$r_x(\varepsilon) = \inf\left\{|x'| : \exists y' \text{ s.t. } (x',y') \in R_{2,x}\right\}.$$

Since the boundary of $R_{2,x}$ is contained in one of the sets $f^{-1}(\varepsilon^2) \cap U_2$ or $\tilde{f}^{-1}(\varepsilon^2) \cap U_2$, it follows that $r_x(\varepsilon)$ satisfies $f(r_x(\varepsilon), \frac{1}{2}r_x(\varepsilon)) = \varepsilon^2$ or $f(r_x(\varepsilon), -\frac{1}{2}r_x(\varepsilon)) = \varepsilon^2$, or $\tilde{f}(r_x(\varepsilon), \frac{1}{2}r_x(\varepsilon)) = \varepsilon^2$, $\tilde{f}(r_x(\varepsilon), -\frac{1}{2}r_x(\varepsilon)) = \varepsilon^2$. It follows that $r_x(\varepsilon)^2 = 2\varepsilon^2(1 + O(\varepsilon))$. Thus on $R_{2,x}$ we have

$$\begin{aligned}
\left|\int_{R_{2,x}} \frac{dx'dy'}{x'y'}\right| &\leq \int_{r_x(\varepsilon)}^{\rho} \frac{dx'}{|x'|}\left|\int_{f^{-1}(\varepsilon^2)}^{\tilde{f}^{-1}(\varepsilon^2)} \frac{dy}{y}\right| \\
&= \int_{r_x(\varepsilon)}^{\rho} \frac{dx'}{|x'|}\left|\log\left|\frac{\tilde{f}_{x'}^{-1}(\varepsilon^2)}{f_{x'}^{-1}(\varepsilon^2)}\right|\right|.
\end{aligned}$$

Using (3.3.2) and (3.3.3), we find that $f_{x'}^{-1}(\varepsilon^2) = \frac{\varepsilon^2}{x'}(1 + O_{x'}(\varepsilon))$ and similarly for $\tilde{f}$. Thus

$$\left|\log\left|\frac{\tilde{f}_{x'}^{-1}(\varepsilon^2)}{f_{x'}^{-1}(\varepsilon^2)}\right|\right| = h_{x'}(\varepsilon),$$

where $h_{x'}(\varepsilon)$ is a continuous function in x' and ε and $h_{x'}(\varepsilon) \to$ as $\varepsilon \to 0$ for each fixed x'.

$$\left| \int_{R_{2,x}} \frac{dx'dy'}{x'y'} \right| \leq \int_{r_x(\varepsilon)<|x|<\rho} \frac{dx'}{|x'|} h_{x'}(\varepsilon).$$

Since $h_{x'}(\varepsilon)/\varepsilon$ has a limit as $\varepsilon \to 0$ pointwise in x' and $r_x(\varepsilon) \sim \varepsilon$, we get that $h_{x'}(\varepsilon)/r_x(\varepsilon)$ is bounded by a constant C (independent of ε) for $0 \leq |x| \leq \rho$. Fix $\delta > 0$. We then get

$$\begin{aligned} \int_{r_x(\varepsilon)<|x|<\rho} \frac{dx'}{|x'|} h_{x'}(\varepsilon) &\leq \int_0^\delta \frac{h_{x'}(\varepsilon)}{r_x(\varepsilon)} dx' + \int_\delta^\rho \frac{dx'}{|x'|} h_{x'}(\varepsilon) \\ &\leq \delta C + \int_\delta^\rho \frac{dx'}{|x'|} h_{x'}(\varepsilon) \\ &\leq \delta C + \frac{1}{\delta} \int_\delta^\rho h_{x'}(\varepsilon) dx. \end{aligned}$$

Letting $\varepsilon \to 0$ first and noticing that $h_{x'}(\varepsilon) \to 0$ pointwise, we find that $\int_\delta^\rho h_{x'}(\varepsilon) dx' \to 0$. Letting now $\delta \to 0$ we find that

$$\left| \int_{R_{2,x}} \frac{dx'dy'}{x'y'} \right| \to 0$$

as $\varepsilon \to 0$.

The estimate for the integral over $R_{2,y}$ is obtained by exchanging the roles of x' and y'.

Thus the integral in the definition of $V(\pi)$ is independent of the choice of f. Taking $f = x^2 - y^2$, we easily obtain that the limit exists; indeed, one can reduce the question to the case that U is a disk of radius ε, in which case the two integrals in the definition of $V(\pi)$ are equal (with opposite signs) by symmetry.

Proposition 3.3.5. *Let π_1 and π_2 be Poisson structures on two open disk neighborhoods U_1 and U_2 of $(0,0) \in \mathbb{R}^2$, given by*

$$\pi_\alpha = \lambda_\alpha (x^2 - y^2) \partial_x \wedge \partial_y |_{U_\alpha}, \quad \alpha = 1,2.$$

Then there exists a Poisson isomorphism $\theta : (U_1, \pi) \to (U_2, \pi_2)$ if and only if $\lambda_1 = \lambda_2$, $R_j(\pi_1) = R_j(\pi_2)$, $j = 1,2$ and $V(\pi_1) = V(\pi_2)$. Moreover, if these conditions are satisfied, the isomorphism can be chosen to extend to all of $\mathbb{R}^2$ and be identity outside any prescribed open set containing U_1 and U_2.

Proof. The necessity follows from the fact that $\lambda_1, \lambda_2, R_1, R_2$ and $V(\pi)$ are invariants.

To prove sufficiency, we proceed as follows. Assume that the intersections of U_1 with ℓ_1 are at $(\alpha_\pm, \alpha_\pm)$ and the intersection with ℓ_2 are at $(\beta_\pm, -\beta_\pm)$. Thus $R_1(\pi_1) = \log\left(-\frac{\alpha_+}{\alpha_-}\right)$, $R_2(\pi_1) = \log\left(-\frac{\beta_+}{\beta_-}\right)$. By replacing (U_1, π_1) with $F(t)(U_1, \pi_1)$, where $F(t)$ is the flow at time t of the modular vector field of π_1 with respect to the area form $\frac{1}{2}dx_1 \wedge dy_1$, we get a Poisson-isomorphic neighborhood, for which the points of intersection are changed to $(e^{\lambda_1 t}\alpha_\pm, e^{\lambda_1 t}\alpha_\pm)$ for the ℓ_1-intersection and $(e^{-\lambda_1 t}\beta_\pm, -e^{-\lambda_1 t}\beta_\pm)$ for the ℓ_2-intersection. By further replacing $F(t)(U_1,\pi_1)$ with its image under the dilation map $(x,y) \mapsto (rx,ry)$, we can change the intersections to $(re^{\lambda_1 t}\alpha_\pm, re^{\lambda_1 t}\alpha_\pm)$ and $(re^{-\lambda_1 t}\beta_\pm, -re^{-\lambda_1 t}\beta_\pm)$, respectively. Denote the corresponding intersections for (U_2,π_2) by $\alpha'_\pm$ and $\beta'_\pm$, respectively. The assumption that $R_j(\pi_1) = R_j(\pi_2),\ j = 1,2$ implies that

$$\log\left(-\frac{\alpha_+}{\alpha_-}\right) = \log\left(-\frac{\alpha'_+}{\alpha'_-}\right)$$

and

$$\log\left(-\frac{\beta_+}{\beta_-}\right) = \log\left(-\frac{\beta'_+}{\beta'_-}\right).$$

It follows that by choosing t and r appropriately, we may assume $\alpha_\pm = \alpha'_\pm$ and $\beta_\pm = \beta'_\pm$.

Now choose an open disk $O \subset U_1 \cap U_2$. Since the lengths of the zero curves of π_1 and π_2 on the complement of O in U are the same, and (by assumptions on the regularized Liouville volumes) the regularized volumes of $U_i \setminus O$ are equal. We can now argue exactly as in the proof of Lemma 3.3.1 to extend the identity map on O and $\mathbb{R}^2 \setminus W$ for an open neighborhood W of $U_1 \cup U_2$ to the desired isomorphism. □

3.3.6 A complete set of invariants in the quadratic case.

Returning now to Poisson structures on compact oriented surfaces, assume that π_1 and π_2 are Poisson structures on Σ_1 and Σ_2, so that π_α vanish linearly on their zero sets $Z(\pi_\alpha)$ except possibly at some special points $p^1_\alpha, \dots, p^m_\alpha \in \Sigma_\alpha$, $\alpha = 1,2$. Assume furthermore that in a neighborhood U^j_α of p^j_α, π_α has the form $\pi_\alpha(x,y)|_{U^j_\alpha} = \lambda^j_\alpha(x^2 + \sigma^j_\alpha y^2)\partial_x \wedge \partial_y$, $\alpha = 1,2$, where $\lambda^j_\alpha \neq 0$ and $\sigma^j_\alpha = \pm 1$.

Let $\Sigma'_\alpha = \Sigma_\alpha \setminus (\bigsqcup^m_{j=1} U^j_\alpha)$. Define the regularized Liouville volume invariant of π_α to be the sum

$$V(\pi_\alpha) \doteq \Sigma^m_{j=1} V\left(\pi_\alpha|_{U^j_\alpha}\right) + V\left(\pi_\alpha|_{\Sigma'_\alpha} : \Sigma'_\alpha \subset \Sigma_\alpha\right),$$

where we set for convenience $V\left(\pi_\alpha|_{U^j_\alpha}\right) = \hat{\infty}$ (a special symbol) if $\sigma^j_\alpha = +1$, and we set $r + \hat{\infty} = \hat{\infty}$ if $r \in \mathbb{R} \cup \{\pm\infty\}$.

We now consider the zero set $Z(\pi_\alpha)$. Embed Σ_α into the three-dimensional space $\mathbb{R}^3$, and replace the zero set $Z(\pi_\alpha) \subset \Sigma_\alpha$ by a set $C(\pi_\alpha)$ of closed curves in $\mathbb{R}^3$ such that:

- $C(\pi_\alpha)|_{\Sigma_\alpha \setminus \bigsqcup^m_{j=1} U^j_\alpha} = Z(\pi_\alpha)|_{\Sigma_\alpha \setminus \bigsqcup^m_{j=1} U^j_\alpha}$;

- If p_α^j is an elliptic point ($\sigma_\alpha^j = 1$), then $C(\pi_\alpha)|_{U_\alpha^j} = Z(\pi_\alpha)|_{U_\alpha^j}$;
- If p_α^j is a hyperbolic point ($\sigma_\alpha^j = -1$), then $C(\pi_\alpha)|_{U_\alpha^j}$ consists of two non-intersecting curves obtained by resolving the intersection of the sets Z^+ and Z^-.

This procedure is illustrated by the diagram below

Assign to each resulting closed curve $\gamma \subset C(\pi_\alpha)$ its modular period $I_\gamma(\pi)$ by adding the modular flow times between the exceptional points on γ.

Theorem 3.3.6. *With the above notation and assumptions, let $\phi : \Sigma_1 \to \Sigma_2$ be a diffeomorphism, so that $\phi(Z(\pi_1)) = Z(\pi_2)$ and $\phi(p_1^i) = \phi(p_2^i)$, $i = 1, \ldots, m$. Assume that the map $Z(\pi_1) \to Z(\pi_2)$ induces the map $\phi^C : C(\pi_1) \to C(\pi_2)$. Then there exists a Poisson isomorphism $\theta : \Sigma_1 \to \Sigma_2$, $\theta_*\pi_1 = \pi_2$ such that $\theta|_{Z(\pi_1)} = \phi^C$ if and only if*

1. $I_\gamma(\pi_1) = I_{\phi^C(\gamma)}(\pi_2)$ *for all* $\gamma \in C(\pi_1)$;
2. $V(\pi_1) = V(\pi_2)$;
3. $\lambda_1^i = \lambda_2^i$, $\sigma_1^i = \sigma_2^i$ *for all* $i = 1, \ldots, m$;

Proof. Clearly, $\gamma \mapsto I_\gamma(\pi_\alpha)$ and $V(\pi_\alpha)$ are invariants, and hence the conditions are necessary.

Conversely, assume that the invariants are the same. Choose a curve $\gamma \in C(\pi_1)$. Choose points a_1^i, b_1^i on γ so that $p_1^i \in [a_1^i, b_1^i] \subset \gamma$ for all exceptional points p_1^i on γ. Do the same for the corresponding curve in $C(\pi_2)$, calling the points a_2^i, b_2^i. The assumption (1) guarantees that this can be done in such a way that the flow times along the corresponding segments are equal, $I(a_1^i, b_1^i) = I(a_2^i, b_2^i)$ and $I(a_1^i, b_1^{i+1}) = I(a_2^i, b_2^{i+1})$. Now choose disjoint contractible neighborhoods $U_\alpha^i \subset \Sigma_\alpha$ of p_α^i so that $V(U_1^i) = V(U_2^i)$ for all i. Using the results of the previous section, we can now find Poisson isomorphisms of U_1^i and U_2^i for all i. Similar procedure should be done for all pair of the corresponding curves in $C(\pi_1), C(\pi_2)$. The assumption (2) guarantees that $V(\Sigma_1 \setminus (U_1^1 \cup \cdots \cup U_1^m)) = V(\Sigma_2 \setminus (U_2^1 \cup \cdots \cup U_2^m))$. One then argues as in Moser's theorem that these isomorphisms can be extended to the desired global isomorphism ϕ. ☐

Example 3.3.7. As an example, we discuss the classifying invariants in the case of a Poisson structure on a 2-torus, given by $\pi_h = h \cdot \pi'$, where π' is the (nowhere zero) Poisson structure coming from

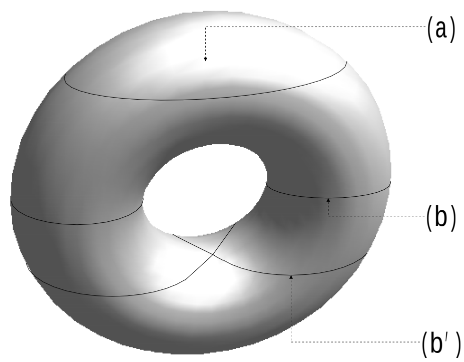

Figure 3.3.2: Poisson structures on a torus.

the area form on the torus, and h is a height function (see Figure 3.3.2). The function h has four critical points, p_1, p_2, p_3, p_4; let's say $h(p_1) < h(p_2) < h(p_3) < h(p_4)$. The points p_1 and p_4 are extremal, and p_2 and p_3 are saddle points.

The zero set of the structure π_h is the set of zeros of h. If h is nowhere zero on the torus (i.e., $h(p_1) > 0$ or $h(p_4) < 0$), π comes from a symplectic structure, and is determined up to isomorphism by the associated Liouville volume.

Generically, h is nonzero at its critical points. Then we have the following possibilities:

(a) $h(p_1) < 0 < h(p_2)$ or $h(p_3) < 0 < h(p_4)$ (the latter case is shown on the figure). The zero set of π_h consists of a single curve. The structure is determined by two invariants: the modular period around the curve and the regularized Liouville volume.

(b) $h(p_2) < 0 < h(p_3)$. The zero set of π_h consists of two ellipses (shown on the figure). There are now three invariants: the two modular periods around each of the curves, and the regularized Liouville volume.

In addition, h could be zero at one of its critical points. There are two possibilities:

(a') $h(p_1) = 0$ or $h(p_4) = 0$ (not shown). In this case, the zero set of π_h is a single point. The structure is determined by one invariant, the modular eigenvalues at that point. It is worth noting that if we replace h by $h_\varepsilon = h \pm \varepsilon$, then, for a suitable choice of the sign, π_{h_ε} will be as in (a). As $\varepsilon \to 0$, the total volume of π_{h_ε} becomes infinite. The modular eigenvalue invariant of π_h can be recovered from the behavior of the modular period of π_{h_ε} as $\varepsilon \to 0$. Indeed, introduce local coordinates x, y near the critical point of h; assume that $h(x,y) = \lambda(x^2 + y^2)$, and $\pi' = \partial_x \wedge \partial_y$. The modular vector field X of π_{h_ε} does not depend on ε and is given by $\pm 2\lambda(x\partial_y - y\partial_x)$ (the sign depends on whether we are at p_1 or p_4). Let γ_ε be the zero curve of π_{h_ε}. The curve is a circle of radius $r = \sqrt{\varepsilon/|\lambda|}$. Then the lengths of $X|_{\gamma_\varepsilon}$ is constant and equals $2|\lambda| r$, and the length of the curve is $2\pi r$. Thus the modular period is $\pi/|\lambda|$, and its limit as $\varepsilon \to 0$ allows us to compute $|\lambda|$ (which is the modular eigenvalues invariant for π_h).

(b') $h(p_2) = 0$ or $h(p_3) = 0$ (shown in the picture). In this case, the zero set of π_h is a "figure 8 curve". The structure is determined by 3 invariants: the modular eigenvalue invariant, the regularized modular period, and the regularized total volume. Again, we can perturb h by setting $h_\varepsilon = h \pm \varepsilon$. For a suitable choice of the sign, π_{h_ε} will be as in (b). The three classifying invariants for π_{h_ε} give rise to the three classifying invariants for π_h. The regularized volume for π_h is the limit of the regularized Liouville volume for π_{h_ε}. To obtain the other invariants, consider coordinates (x,y) near the critical point of h, so that $h(x,y) = \lambda(x^2 - y^2)$ and $\pi' = \partial_x \wedge \partial_y$. The modular vector field X in these coordinates is (up to sign) $\pm 2\lambda(x\partial_y + y\partial_x)$. Let $U = \{(x,y) : x^2 + y^2 < R\}$. Denote by $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ the zero curves of π_{h_ε}. Then the modular period $T_{\gamma_j}(\pi_{h_\varepsilon})$ around γ_j can be computed as

$$T_{\gamma_j}(\pi_{h_\varepsilon}) = \int_{\gamma_j(\varepsilon)\cap U} \frac{1}{|X|} d\gamma_j + \int_{\gamma_j(\varepsilon)\setminus U} \frac{1}{|X|} d\gamma_j.$$

The second integral has a finite limit as $\varepsilon \to 0$, which is the modular length of a portion of the zero curve of π_h lying outside of U and to the right or to the left of p, depending on j. The first integral can be computed as

$$\begin{aligned} f_j(\varepsilon) = \int_{-R}^{R} \frac{1}{2\lambda\sqrt{x^2+y^2}} \cdot \sqrt{1 + \frac{x^2}{y^2}} dx &= \frac{1}{2\lambda} \int_{\cosh^{-1}(-R/\delta)}^{\cosh^{-1}(R/\delta)} dt \\ &\sim \frac{1}{\lambda} \log(R/\delta) = \text{const} - \frac{1}{\lambda} \log \varepsilon, \end{aligned}$$

where $\delta = \varepsilon/\lambda = x^2 - y^2$. It follows that

$$\lambda = \lim_{\varepsilon \to 0} \frac{\log \varepsilon}{T_{\gamma_j}(\pi_{h_\varepsilon})}$$

(the limit is independent of j); this expresses the modular eigenvalues invariant in terms of the behavior of $T_{\gamma_j}(\pi_{h_\varepsilon})$. Moreover, the limit

$$\lim_{\varepsilon\to 0}(T_{\gamma_1}(\pi_{h_\varepsilon}) - T_{\gamma_2}(\pi_{h_\varepsilon}))$$

is equal to

$$\lim_{\varepsilon\to 0}\left(\int_{\gamma_1(\varepsilon)\setminus U}\frac{1}{|X|}d\gamma_1 - \int_{\gamma_2(\varepsilon)\setminus U}\frac{1}{|X|}d\gamma_2\right)$$

(independently of U, since modifying R increases the two integrals by the same amount, not changing their differences). This limit gives the regularized modular flow time around the zero curve of π_h.

Chapter 4

Gauge and Morita Equivalence of Poisson manifolds.

4.1 Summary of results

In this Chapter, we compute the effect of a gauge transformation on a symplectic groupoid of an integrable Poisson manifold. We also show that gauge equivalence of integrable Poisson manifolds implies their Morita equivalence, thus relating the two notions of equivalence. To obtain this result, we first prove that gauge transformations of Poisson structures are compatible with (anti-)Poisson maps. As an example, we classify topologically stable Poisson structures on a two-sphere up to Morita equivalence. To do so, we first prove that the modular periods around the zero curves with linear degeneracy are invariant under Morita equivalence, and that the leaf spaces of Morita equivalent Poisson manifolds are homeomorphic. On the other hand, using the results on classification of structures with linear degeneracies obtained earlier, we show that if two topologically stable structures on a compact oriented surface have the same modular period invariants, but possibly different regularized volumes, they are gauge equivalent. It remains then to note that the topologically stable structures are integrable to conclude that they are Morita equivalent. These results are a part of the joint work [BR] with H. Bursztyn. In this chapter, we restrict ourselves to Poisson structures, rather than the more general Dirac structures; see [BR] for some results on gauge transformations of Dirac structures.

In this Chapter, (P_1,π_1) and (P_2,π_2) denote Morita-equivalent Poisson manifolds with a Morita-equivalence bimodule (S,Ω) so that we have a diagram $(P_1,\pi_1) \overset{J_1}{\leftarrow} (S,\Omega) \overset{J_2}{\rightarrow} (P_2,\pi_2)$.

4.2 Properties of Morita equivalence

For Poisson manifolds, symplectic realizations are an analog of representations for associative algebras. A symplectic realization $\alpha : (S,\Omega) \to (P,\pi)$ is called complete if α is complete as a Poisson map, i.e., the pull-back of a compactly supported function on P has a complete Hamiltonian vector field on S. The *"category" of complete symplectic realizations* (introduced in [Xu91b]) for a given Poisson manifold P is the "category" with objects being the complete symplectic realizations of P and the morphisms from a realization $(S_1,\Omega_1) \xrightarrow{\alpha_1} (P,\pi)$ to a realization $(S_2,\Omega_2) \xrightarrow{\alpha_2} (P,\pi)$ being lagrangian submanifolds in $(S_2,\Omega_2) *_P (S_1,-\Omega_1) = (\{(x,y) \in S_2 \times S_1 | \, \alpha_1(x) = \alpha_2(y)\}, \Omega_2 \times -\Omega_1)$. This is not a true category, because a certain transversality assumption is necessary for composition of two morphisms to be defined (cf. [Xu91a]). It turns out ([Xu91a]) that the main characterizing property of Morita equivalence for algebras has an analog in the case of Poisson manifolds. Namely, Morita equivalent Poisson manifolds have equivalent "categories" of complete symplectic realizations.

It is a natural problem to classify Poisson manifolds up to Morita equivalence. The answer is simple in the case of symplectic manifolds: the fundamental group of a manifold is the complete invariant of Morita equivalence. A generalization of this result to a certain class of regular Poisson manifolds was obtained in [Xu91a]. However, much work is still to be done to answer this question in more general situations. Finding various invariants of Morita equivalence can be considered as a first step in this direction.

4.2.1 Invariance of the topology of the leaf space.

In this subsection we prove that the leaf spaces of Morita equivalent Poisson manifolds are homeomorphic as topological spaces.

Let (P,π) be a Poisson manifold. Let

$$L(P) \doteq P/\{x \sim y \text{ if } x \text{ and } y \text{ are in the same leaf}\}$$

be the leaf space of P. Let $p : P \to L(P)$ be the quotient map. Endow $L(P)$ with its quotient topology: a function $f : L(P) \to X$ valued in a topological space X is continuous iff $f \circ p : P \to X$ is continuous as a function on P.

Let (P_1,π_1) and (P_2,π_2) be Poisson manifolds and $(P_1,\pi_1) \xleftarrow{J_1} (S,\Omega) \xrightarrow{J_2} (P_2,\pi_2)$ be their Morita equivalence bimodule. It is well-known (see e.g. [CW99]) that S induces a bijection of sets

$\phi_S : L(M_1) \to L(M_2)$ given by

$$\phi_S(\mathscr{L}) = J_2(J_1^{-1}(\mathscr{L})) \tag{4.2.1}$$

for all leaves $\mathscr{L} \in L(P_1)$.

Proposition 4.2.1. *The map $\phi_S : L(P_1) \to L(P_2)$ is a homeomorphism of topological spaces.*

Proof. Let F_i, $i = 1,2$ be the subset of TM_i consisting of vectors tangent to the symplectic leaves. Let $TJ_i \subset TS$, $i = 1,2$ be the subbundles tangent to the J_i-fibers. Then

$$J_1^* F_1 = J_2^* F_2 = TJ_1 + TJ_2 \doteq F,$$

where $J_i^* F_i = \{v \in TS | \, TJ_i v \in F_i\}$ denotes the pull-back of F_i.

Since the fibers of J_i, $i = 1,2$, are connected, the natural maps

$$\psi_i : S/F \to M_i/F_i$$

of leaf spaces are bijections. Moreover, it is not hard to see that ϕ_S is obtained as

$$\phi_S = \psi_2 \circ \psi_1^{-1} : M_1/F_1 \to M_2/F_2.$$

Endow S/F with its quotient topology. Then it is sufficient to prove that ψ_i, $i = 1,2$ are homeomorphisms.

By definition of the quotient topology, the map $\psi_i : S/F \to M_i/F_i$ is continuous iff the map $\psi_i \circ p : S \to M_i/F_i$ is continuous (here $p : S \to S/F$ is the quotient map). But $\psi_i = p_i \circ J_i$, where $p_i : M_i \to M_i/F_i$ is the quotient map. Hence ψ_i is continuous.

Similarly, $\psi_i^{-1} : M_i/F_i \to S/F$ is continuous iff $\psi_i^{-1} \circ p_i : M_i \to S/F$ is continuous. Since J_i is a submersion, this is true iff $\psi_i^{-1} \circ p_i \circ J_i : S \to S/F$ is continuous. But $\psi_i^{-1} \circ p_i \circ J_i = p$. Therefore, ψ_i is a homeomorphism for $i = 1,2$, which implies that ϕ_S is a homeomorphism. $\square$

4.2.2 Invariance of the modular class.

Here we recall (see [Cra] and [Gin] for the details) that the modular class of a Poisson manifold is preserved under Morita equivalence.

By a result of Ginzburg and Lu [GL92], the isomorphism (4.2.1) produces an isomorphism of Poisson cohomologies

$$\phi_S^* : H^1_\pi(P_1) \to H^1_\pi(P_2). \tag{4.2.2}$$

By a theorem of Crainic [Cra] and Ginzburg [Gin], this isomorphism turns out to preserve the modular class:

$$\phi_S^*(\mu_{(P_1,\pi_1)}) = \mu_{(P_2,\pi_2)}.$$

We will need the following remark from the construction of the isomorphism (4.2.2) in [Gin]:

Remark 4.2.2. Given volume forms ν_1 and ν_2 on P_1 and P_2, respectively, there exists a vector field X on S, with the property that $(J_i)_*X = X^{\nu_i}$, $i = 1,2$, where X^{ν_i} is the modular vector field of (P_i,π_i) with respect to ν_i. Such a vector field X on S is actually Hamiltonian, and its Hamiltonian H is determined by the equation

$$DJ_1^*\nu_1 = \pm e^H J_2^*\nu_2,$$

where $D : \Omega^k(M) \to \Omega^{2m-k}(M)$ is the symplectic $*$-operator (cf. [Bry88]).

4.2.3 Invariance of modular periods.

For convenience of the reader and to set the notation, we recall the definition of the modular period invariant given in Section 3.2.4. Suppose that the Poisson tensor π on P vanishes on a closed curve $\gamma \subset P$, and is nonzero away from γ in a neighborhood of γ. Since the modular vector field X^ν of π with respect to ν preserves the Poisson structure, its flow must take the zero set of π to the zero set of π. Thus the flow of X^ν preserves γ and, therefore, X^ν must be tangent to γ. Moreover, for any other choice ν' of the volume form we have

$$X^{\nu'}|_\gamma = X^\nu|_\gamma + (X_{\log\frac{\nu'}{\nu}})|_\gamma = X^\nu|_\gamma,$$

since all Hamiltonian vector fields are zero when restricted to the zero curve γ. It follows that the restriction of the modular vector field X^ν to γ is independent of ν. In particular, as was observed by Roytenberg [Roy], the period of the flow of this vector field around γ is an invariant of the Poisson structure π. We denote this number by $T_\gamma(P,\pi)$ (or, shortly, $T_\gamma(\pi)$ when it is clear what P is).

In order to establish the invariance of a modular period with respect to Morita equivalence, we first prove the following more general result:

Theorem 4.2.3. *Let (P_i,π_i), $i = 1,2$ be Poisson manifolds and $(P_1,\pi_1) \overset{J_1}{\leftarrow} (S,\Omega) \overset{J_2}{\rightarrow} (P_2,\pi_2)$ be a Morita equivalence bimodule. Assume that $Z_i \subset P_i$ are subsets for which $\pi_i|_{Z_i} = 0$ and that the isomorphism of leaf spaces satisfies*

$$\phi_S(Z_1) = Z_2.$$

Let Φ_t^i be the flow of the modular vector field X^{ν_i} for some volume forms ν_i on M_i, $i = 1,2$. Assume that Φ_t^i takes Z_i to Z_i for all t. Then

$$\phi_S \circ \Phi_t^1 = \Phi_t^2 \circ \phi_S, \quad \forall t \in \mathbb{R}.$$

Proof. Let X be a vector field on S such that $(J_i)_* X = X_i^{\nu_i}$, $i = 1,2$ (see Remark 4.2.2) and let Φ_t be its flow. By the definition of ϕ_S, for each point $p_1 \in Z_1$ (which forms by itself a symplectic leaf) we have

$$\phi_S(\{p_1\}) = J_2(J_1^{-1}(\{p_1\}))$$

which by our assumption on ϕ_S is a single point $p_2 \in Z_2$. It follows that $J_1^{-1}(\{p_1\}) \subset J_2^{-1}(\{p_2\})$. Reversing the roles of p_1 and p_2 we get $J_1^{-1}(\{p_1\}) = J_2^{-1}(\{p_2\})$. In particular, it follows that

$$J_1^{-1}(Z_1) = J_2^{-1}(Z_2).$$

Thus for any fixed $r \in J_1^{-1}(\{p_1\}) = J_2^{-1}(\{p_2\})$ we obtain

$$\Phi_t^2(\phi_S(\{p_1\})) = J_2(\Phi_t(\{r\})) = J_2(J_1^{-1}(\Phi_t^1(\{p_1\})) = \phi_S(\Phi_t^1(\{p_1\})).$$

Therefore, $\phi_S \circ \Phi_t^1 = \Phi_t^2 \circ \phi_S$. □

Corollary 4.2.4. *Let (P_1, π_1) and (P_2, π_2) be Morita equivalent Poisson manifolds with an equivalence bimodule (S, Ω). Assume that $\gamma_i \subset P_i$ are simple closed curves, and there exist open sets $U_i \supset \gamma_i$, so that that $\pi_i|_{\gamma_i} = 0$ and $\pi|_{U_i \setminus \gamma_i} \neq 0$, $i = 1,2$. Assume finally that $\phi_S(\gamma_1) = \gamma_2$. Then*

$$T_{\gamma_1}(P_1, \pi_1) = T_{\gamma_2}(P_2, \pi_2).$$

Proof. Applying Theorem 4.2.3 with $Z_i = \gamma_i$, $i = 1,2$ we obtain that the flows Φ_t^i of modular vector fields $X_i^{\nu_i}$ are intertwined by ϕ_S. Thus for any $p_1 \in \gamma_1$

$$\begin{aligned} T_{\gamma_1}(P_1, \pi_1) &= \inf\{t > 0 : \Phi_t^1(p_1) = p_1\} \\ &= \inf\{t > 0 : \phi_S(\Phi_t^1(p_1)) = \phi_S(p_1)\} \\ &= \inf\{t > 0 : \Phi_t^2(\phi_S(p_1)) = \phi_S(p_1)\} \\ &= \inf\{t > 0 : \Phi_t^2(p_2) = p_2\}, \qquad p_2 = \phi_S(p_1) \\ &= T_{\gamma_2}(P_2, \pi_2). \end{aligned}$$

Thus the modular period is a Morita equivalence invariant, as claimed. □

4.2.4 Tangent space to a Morita equivalence bimodule.

In this section we collect several useful statements about the structure of the tangent space at a point of a Morita equivalence bimodule.

Let (P_1,π_1) and (P_2,π_2) be Morita equivalent Poisson manifolds, and let (S,Ω,α,β) be their Morita equivalence bimodule. Denote by $\alpha_x \doteq \alpha^{-1}(\alpha(x))$ the α-fiber through $x\in S$ and by $\beta_x \doteq \beta^{-1}(\beta(x))$ — the β-fiber through $x \in S$. Since $(T_x\alpha_x)^\Omega = T_x\beta_x = \{X_{\alpha^* f}|\, f\in C^\infty(P_2)\}$, $(T_x\beta_x)^\Omega = T_x\alpha_x = \{X_{\beta^* g}|\, g\in C^\infty(P_1)\}$, we have

$$\{\alpha^* C^\infty(P_1), \beta^* C^\infty(P_2)\} = 0$$

Let $\mathscr{L}_{\alpha(x)}$ be the symplectic leaf of P_1 through the point $\alpha(x)\in P_1$ and let $\mathscr{L}_{\beta(x)}$ be the symplectic leaf of P_2 through the point $\beta(x)\in P_2$.

Claim 4.2.5. The tangent space at a point $x\in S$ satisfies the following properties:

1. $T_x\alpha_x/T_x(\alpha_x\cap\beta_x) \simeq T_{\beta(x)}\mathscr{L}_{\beta(x)}$;

2. $T_{\alpha(x)}P_1/T_{\alpha(x)}\mathscr{L}_{\alpha(x)} \simeq T_{\beta(x)}P_2/T_{\beta(x)}\mathscr{L}_{\beta(x)}$;

3. There is the following splitting of T_xS:

$$T_xS \simeq T_{\alpha(x)}\mathscr{L}_{\alpha(x)} \oplus T_{\beta(x)}\mathscr{L}_{\beta(x)} \oplus T_x(\alpha_x\cap\beta_x) \oplus T_{\alpha(x)}P_1/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}; \tag{4.2.3}$$

   In particular, $T_x\alpha_x \simeq T_x(\alpha_x\cap\beta_x)\oplus T_{\beta(x)}\mathscr{L}_{\beta(x)}$ and $T_x\beta_x \simeq T_x(\alpha_x\cap\beta_x)\oplus T_{\alpha(x)}\mathscr{L}_{\alpha(x)}$. With this splitting, we have the following orthogonality relations with respect to the symplectic form

$$\Omega\left(T_{\alpha(x)}\mathscr{L}_{\alpha(x)}, T_{\beta(x)}\mathscr{L}_{\beta(x)}\oplus T_x(\alpha_x\cap\beta_x)\oplus T_{\alpha(x)}P/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}\right) = 0; \tag{4.2.4}$$

$$\Omega\left(T_{\beta(x)}\mathscr{L}_{\beta(x)}, T_{\alpha(x)}\mathscr{L}_{\alpha(x)}\oplus T_x(\alpha_x\cap\beta_x)\oplus T_{\alpha(x)}P/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}\right) = 0; \tag{4.2.5}$$

$$\Omega\left(T_{\alpha(x)}P/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}, T_{\alpha(x)}P/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}\right) = 0; \tag{4.2.6}$$

$$\Omega(T_x(\alpha_x\cap\beta_x), T_x(\alpha_x\cap\beta_x) = 0; \tag{4.2.7}$$

   Moreover, the restrictions $\Omega|_{T_{\alpha(x)}\mathscr{L}_{\alpha(x)}}$ and $\Omega|_{T_{\beta(x)}\mathscr{L}_{\beta(x)}}$ are non-degenerate forms, and Ω gives a non-degenerate pairing of $T_x(\alpha_x\cap\beta_x)$ and $T_{\alpha(x)}P_1/T_{\alpha(x)}\mathscr{L}_{\alpha(x)} \simeq T_{\beta(x)}P_2/T_{\beta(x)}\mathscr{L}_{\beta(x)}$.

Proof. (1) Let $T_x\beta : T_xS \to T_{\beta(x)}P_2$ be the map induced by β on the level of tangent spaces. The image of the restriction $T_x\beta|_{T_x\alpha_x}$ of this map to $T_x\alpha_x$ lies in $T_{\beta(x)}\mathscr{L}_{\beta(x)}$. Since $\ker(T_x\beta) = T_x\beta_x$, there is a well-defined quotient map $\widehat{T_x\beta}|_{T_x\alpha_x} : T_x\alpha_x/T_x(\alpha_x\cap\beta_x) \to T_{\beta(x)}\mathscr{L}_{\beta(x)}$. For $\xi\in T_{\beta(x)}\mathscr{L}_{\beta(x)}$, let $f\in$

$C^\infty(P_2)$ be a function whose hamiltonian vector field X_f has the value ξ at $\beta(x)$, i.e. $X_f(\beta(x)) = \xi$. Then $\widehat{T_x\beta}|_{T_x\alpha_x}(X_{\beta^* f})(x) = \xi$. Therefore, the map $\widehat{T_x\beta}|_{T_x\alpha_x} : T_x\alpha_x/T_x(\alpha_x \cap \beta_x) \to T_{\beta(x)}\mathscr{L}_{\beta(x)}$ is onto.

The second statement of the Claim follows from the isomorphism of leaf spaces of Morita equivalent Poisson manifolds, and the last one follows from the first two. $\square$

4.3 Gauge equivalence of Dirac structures and Poisson manifolds

In this section we recall the definitions of gauge transformations and gauge equivalence of Poisson manifolds, and derive an equivariance property of gauge transformations with respect to (anti)-Poisson maps.

4.3.1 Dirac structures

In order to define gauge transformations of Poisson structures by closed 2-forms, we first need to recall the notion of a Dirac structure, generalizing that of a Poisson structure.

Dirac structures were introduced in [Cou90] to provide a geometric framework for the study of constrained mechanical systems. Examples of Dirac structures include Poisson and pre-symplectic structures, as well as foliations. In general, a Dirac structure determines a singular foliation on a manifold together with a pre-symplectic structure on each leaf of this foliation.

A *linear Dirac structure* on a vector space V is a subspace $L \subset V \oplus V^*$ which is maximally isotropic with respect to the symmetric pairing $\langle\,,\,\rangle$ defined by

$$\langle (x,\omega),\,(y,\nu)\rangle = \frac{1}{2}(\omega(y) + \nu(x)), \qquad (x,\omega),(y,\nu) \in V \oplus V^* \tag{4.3.1}$$

In other words, L is an isotropic subspace and $\dim(L) = \dim(V)$. For example, for a bivector $\pi \in V \wedge V$ on V the graph $L = \text{graph }(\tilde{\pi})$ of the associated linear map $\tilde{\pi} : V^* \to V$ is a Dirac structure.

A *Dirac structure* on a manifold P is a subbundle $L \subset TP \oplus T^*P$ which determines a linear Dirac structure $L_p \subset T_pP \oplus T_p^*P$, $p \in P$ pointwise and satisfies the following integrability condition: the space of sections of L is closed under the *Courant bracket* $[\,,\,] : \Gamma(TP \oplus T^*P) \times \Gamma(TP \oplus T^*P) \to \Gamma(TP \oplus T^*P)$, given by

$$[(X,\omega),\,(Y,\nu)] \doteq \left([X,Y], L_X\nu - L_Y\omega + \frac{1}{2}d(\omega(X) - \nu(Y))\right). \tag{4.3.2}$$

For a Poisson bivector $\pi \in \mathfrak{X}^1(P)$ on P the graph $L = \text{graph}(\tilde{\pi})$ of the associated bundle map $\tilde{\pi} : T^*P \to TP$ is a Dirac structure; the integrability condition in this case is equivalent to $[\pi,\pi] = 0$.

The Courant bracket (4.3.2) does not satisfy the Jacobi identity in general. However, the Jacobi identity does hold for the restriction of this bracket to the sections of a Dirac bundle $L \subset TP \oplus T^*P$. Thus, on L there is a natural Lie algebroids structure, the bracket being given by the restriction of the Courant bracket and the anchor map being the restriction of the natural projection $TP \oplus T^*P \to TP$. The Lie algebroid structure on L determines a foliation on P. It turns out that this foliation is symplectic if and only if $L = \text{graph}\,(\pi)$ for a Poisson structure π on P .

4.3.2 Gauge transformations and gauge equivalence

The notion of gauge equivalence of Dirac structures was introduced in [SW] motivated by the study of the geometry of Poisson structure "twisted" by a closed 3-form.

The additive group of closed 2-forms on a manifold acts on the set of Dirac structures on the manifold as follows. For a Dirac structure L on P and a closed 2-form $B \in \Omega^2(P)$, define the *gauge transformation* of L by B according to

$$\tau_B(L) \doteq \{(X, \eta + \tilde{B}(X)) \,|\, (X, \eta) \in L\}. \tag{4.3.3}$$

This is equivalent to adding the pull-back of B to the pre-symplectic form on each of the leaves of the foliation defined by L. Two Dirac structures on P which are in the same orbit of the action by gauge transformations are called *gauge-equivalent*.

For a Poisson structure π on P, let $L_\pi = \text{graph}(\tilde{\pi})$ be the corresponding Dirac structure. As was observed in [SW], $\tau_B(L_\pi)$ corresponds to another Poisson structure if and only if the endomorphism $1 + \tilde{B} \circ \tilde{\pi} : T^*P \to T^*P$ is invertible. If this is the case, the Poisson structure π_B such that

$$\tilde{\pi}_B = \tilde{\pi} \circ (1 + \tilde{B} \circ \tilde{\pi})^{-1} \tag{4.3.4}$$

is said to be obtained from π by a gauge transformation. For short, we write $\pi_B = \tau_B(\pi)$ instead of $L_{\pi_B} = \tau_B(L_\pi)$.

Gauge-equivalent Dirac structures have a lot of common properties. For instance, the leaf decomposition is the same (though the pre-symplectic forms on leaves differ by the pull-backs of the closed 2-form defining the transformation). Gauge-equivalent Poisson structures have isomorphic Lie algebroids, and, therefore, isomorphic Poisson cohomology.

4.3.3 Equivariance property of (anti)-Poisson maps with respect to gauge transformations

In this section we prove that Poisson maps are equivariant with respect to gauge transformations of Poisson structures. More generally, one can prove (see [BR]) that Dirac maps (a certain generalization of Poisson maps to the class of Dirac structures) are equivariant with respect to gauge transformations of Dirac structures.

Theorem 4.3.1. *Let (P,π^P) and (Q,π^Q) be Poisson manifolds and let $\varphi:(Q,\pi^Q)\to(P,\pi^P)$ be a Poisson map. Let B be a closed 2-form on P such that the operators $(id+\tilde{B}\circ\widetilde{\pi^P}):T^*P\to T^*P$ and $(id+\widetilde{\varphi^*B}\circ\widetilde{\pi^Q}):T^*Q\to T^*Q$ are invertible (so that $\tau_B\pi^P$ and $\tau_{\varphi^*B}\pi^Q$ define Poisson structures on P and Q respectively). Then $\varphi:\left(Q,\tau_{\varphi^*B}\pi^Q\right)\to\left(P,\tau_B\pi^P\right)$ is also a Poisson map.*

Proof. Since $\varphi:(Q,\pi^Q)\to(P,\pi^P)$ is a Poisson map, we have

$$T_x\varphi\circ\widetilde{\pi^Q_x}\circ T^*_{\varphi(x)}\varphi=\widetilde{\pi^P_{\varphi(x)}},\quad x\in Q, \tag{4.3.5}$$

where $T_x\varphi:T_xQ\to T_xP$ and $T^*_x\varphi:T^*_xP\to T^*_xQ$ are the maps associated to $\varphi:Q\to P$. To prove that $\varphi:(Q,\tau_{\varphi^*B}\pi^Q)\to(P,\tau_B\pi^P)$ is a Poisson map, it suffices to check that the following equality

$$T_x\varphi\circ\widetilde{\tau_{\varphi^*B}\pi^Q_x}\circ T^*_{\varphi(x)}\varphi=\widetilde{(\tau_B\pi^P)}_{\varphi(x)}, \tag{4.3.6}$$

where

$$\widetilde{\tau_B\pi^P}=\widetilde{\pi^P}\cdot\left(\mathrm{id}+\tilde{B}\circ\widetilde{\pi^P}\right)^{-1},$$
$$\widetilde{\tau_{\varphi^*B}\pi^P}=\widetilde{\pi^Q}\cdot\left(\mathrm{id}+\widetilde{\varphi^*B}\circ\widetilde{\pi^Q}\right)^{-1}.$$

Let $O_P\doteq\tilde{B}\circ\widetilde{\pi^P}$ and $O_Q\doteq\widetilde{\varphi^*B}\circ\widetilde{\pi^Q}$ be the operators on T^*P and T^*Q respectively.

Lemma 4.3.2. *The map $T^*_{\varphi(x)}\varphi:T^*_{\varphi(x)}P\to T^*_xQ$ intertwines $(O_P)_{\varphi(x)}$ and $(O_Q)_x$, i.e.*

$$T^*_{\varphi(x)}\varphi\circ(O_P)_{\varphi(x)}=(O_Q)_x\circ T^*_{\varphi(x)}\varphi. \tag{4.3.7}$$

Proof. Let $\chi\in T_yQ$, $\eta\in T^*_{\varphi(y)}P$. We have

$$\begin{aligned}&\langle O_Q\circ\varphi^*(\eta),\chi\rangle=\langle\widetilde{\varphi^*B}\circ\widetilde{\pi^Q}\circ\varphi^*(\eta),\chi\rangle\overset{(I)}{=}\\&=\langle\tilde{B}\circ\varphi^*\circ\widetilde{\pi^Q}\circ\varphi^*(\eta),\varphi_*(\chi)\rangle\overset{(II)}{=}\langle\tilde{B}\circ\widetilde{\pi^P}(\eta),\varphi_*(\chi)\rangle=\\&\langle\varphi^*\tilde{B}\circ\widetilde{\pi^P}(\eta),\chi\rangle=\langle\varphi^*\circ O_P(\eta),\chi\rangle,\end{aligned}$$

where step (I) follows from $\widetilde{\varphi^*B}(\chi)=\tilde{B}(\varphi_*(\chi))$ (which is an easy consequence of the definitions) and step (II) follows from (4.3.5). The claim follows. ☐

Lemma 4.3.3. *The map $T^*\varphi$ intertwines $(id+O_Q)^{-1}$ and $(id+O_P)^{-1}$, i.e.*

$$(id+O_Q)_x^{-1}\circ T^*_{\varphi(x)}\varphi = T^*_{\varphi(x)}\varphi\circ(id+O_P)^{-1}_{\varphi(x)}\,. \tag{4.3.8}$$

Proof. To obtain (4.3.8), add $T^*_{\varphi(x)}\varphi$ to both sides of (4.3.7), and multiply the resulting identity by $(\mathrm{id}+O_Q)_x^{-1}$ on the left, and by $(\mathrm{id}+O_P)_x^{-1}$ on the right. □

Finally, (4.3.6) follows easily from (4.3.8) and (4.3.5). □

To obtain an analogous statement for anti-Poisson maps, we need the following observation:

Lemma 4.3.4. *Let (P,π) be a Poisson manifold and B be a closed 2-form on P such that $(id+\tilde{B}\circ\tilde{\pi})$ is invertible. Then*

$$-\tau_B\pi = \tau_{-B}(-\pi).$$

Theorem 4.3.1 and Lemma 4.3.4 now imply

Theorem 4.3.5. *Let (P,π^P) and (Q,π^Q) be Poisson manifolds and $\varphi:(Q,\pi^Q)\to(P,\pi^P)$ be an anti-Poisson map. Let B be a closed 2-form on P such that the operators $(id+\tilde{B}\circ\widetilde{\pi^P}):T^*P\to T^*P$ and $(id+\widetilde{\varphi^*B}\circ\widetilde{\pi^Q}):T^*Q\to T^*Q$ are invertible. Then $\varphi:\big(Q,\tau_{-(T^*\varphi)B}\pi^Q\big)\to\big(P,\tau_B\pi^P\big)$ is also an anti-Poisson map.*

4.4 Morita equivalence of gauge-equivalent integrable Poisson structures

In this section we compute the result of a gauge transformation of an integrable Poisson manifold on the symplectic structure of its symplectic groupoid and prove that two integrable gauge-equivalent Poisson structures are Morita equivalent.

Let (P,π) be a Poisson manifold and (S,Ω,α,β) be its symplectic groupoid. Let (P,π_B) be the Poisson manifold obtained by a gauge transformation of the original one. Since the Lie algebroids of (P,π) and (P,π_B) are isomorphic, the Lie algebroid of (P,π_B) can be integrated to a Lie groupoid isomorphic to (S,α,β). It is natural to ask the following questions:

1. Is there a symplectic form on S making it into a symplectic groupoid of (P,π_B);

2. Are the manifolds (P,π) and (P,π_B) Morita equivalent?

To answer these questions, we prove the following

Theorem 4.4.1. *Let (P,π) be an integrable Poisson manifold and (S,Ω,α,β) be its symplectic groupoid with connected simply-connected α-fibers. Let $B \in \Omega^2(P)$, $dB = 0$ be a closed 2-form on P such that $\pi_B \doteq \tau_B(\pi)$ is a Poisson structure gauge-equivalent to π. Then*

1. *$S_B \doteq (S,\hat{\Omega},\alpha,\beta)$, where $\hat{\Omega} \doteq \Omega + \alpha^* B - \beta^* B$, is a symplectic groupoid of (P,π_B);*

2. *The Poisson manifolds (P,π) and (P,π_B) are Morita equivalent, with Morita equivalence bimodule $(S,\Omega_B,\alpha,\beta)$, where Ω_B is given by*

$$\Omega_B \doteq \Omega - \beta^* B. \tag{4.4.1}$$

Proof. 1. To prove the first statement, we have to check that $\hat{\Omega}$ is symplectic, that the graph of the groupoid multiplication $\Gamma_m = \{(x,y,m(x,y)) \,|\, (x,y) \in S_2\}$ (where S_2 is the set of composable pairs) is lagrangian in $S_B \times S_B \times \overline{S_B}$, and that the maps $\alpha : S_B \to P$ and $\beta : S_B \to P$ are Poisson and anti-Poisson respectively. The proof of the fact that $\hat{\Omega}$ is symplectic is analogous to the proof that Ω_B is symplectic given below.

Let $(x,y) \in S_2$. Consider a curve $(x(t),y(t))$ in S_2 with $(x(0),y(0)) = (x,y)$. Let $(u,v) = (x'(0),y'(0))$. Then $\left(u,v,\left(T_{(x,y)}m\right)(u,v)\right) \in T_p\Gamma_m$, $p = (x,y,x*y)$ and any tangent vector in $T_p\Gamma_m$ is of this form. Differentiating the identities $\alpha(m(x,y)) = \alpha(x)$, $\beta(m(x,y)) = \beta(y)$ and $\beta(x) = \alpha(y)$, we obtain

$$\begin{aligned}
&T\alpha(Tm(u,v)) = T\alpha(u),\\
&T\beta(Tm(u,v)) = T\beta(v),\\
&T\beta(u) = T\alpha(v).
\end{aligned}$$

Therefore, for $w_i \doteq (u_i,v_i,Tm(u_i,v_i)) \in T_p\Gamma_m$, $i = 1,2$ we have

$$\begin{aligned}
(\hat{\Omega},\hat{\Omega},-\hat{\Omega})(w_1,w_2) \quad = \quad & B(T\alpha(u_1),T\alpha(u_2)) - B(T\beta(u_1),T\beta(u_2)) + \\
& B(T\alpha(v_1),T\alpha(v_2)) - B(T\beta(v_1),T\beta(v_2)) - \\
& -B(T\alpha Tm(u_1,v_1),T\alpha Tm(u_2,v_2)) + \\
& B(T\beta(u_1,v_1),T\beta Tm(u_2,v_2)) = 0.
\end{aligned}$$

Hence, $\Gamma_m \subset S_B \times S_B \times \overline{S_B}$ is lagrangian, and, therefore, S_B is a symplectic groupoid. It is easy to see that α is a Poisson map. Since there is a unique Poisson structure on the identity section of a symplectic groupoid with this property, the Poisson structure induced by $\hat{\Omega}$ on P is π_B.

2. To prove the second part of the theorem, we have to check that $(P,\pi) \xleftarrow{\alpha} (S,\Omega_B) \xrightarrow{\beta} (P,\pi_B)$ satisfies all the properties of a Morita equivalence bimodule. First, we will need the following

Claim 4.4.2. The 2-form $\Omega_B = \Omega - \beta^* B$ on S is symplectic.

Proof. Let $u \in T_x S$ be such that $\Omega_B(u,v) = 0$ for all $v \in T_x S$. Suppose that $v \in T_x\beta_x$. Then $\beta^* B(u,v) = 0$ and therefore, $\Omega_B(u,v) = \Omega(u,v) = 0$. Hence, $u \in (T_x\beta_x)^\Omega = T_x\alpha_x$.

We will now use the splitting (4.2.3) and the orthogonality relations (4.2.4)–(4.2.7). Let $u = u_1 + u_2$, where $u_1 \in T_x(\alpha_x \cap \beta_x)$ and $u_2 \in T_{\beta(x)}\mathscr{L}_{\beta(x)}$.

Suppose that $u_2 \neq 0$. Let ω_B be the symplectic form on the leaf through $\beta(x)$ corresponding to the Poisson structure $\tau_B\pi$. Since $\beta : (S,\Omega) \to (P,\pi)$ is an anti-Poisson map and the form ω (on $\mathscr{L}_{\beta(x)}$) corresponding to the Poisson structure is non-degenerate, there exists $v \in T_{\beta(x)}\mathscr{L}_{\beta(x)} \subset T_x\beta_x$ such that $\Omega(u_2,v) \neq 0$. Since $\beta^* B(T_x\beta_x,\cdot) = 0$, $\Omega_B(u,v) = \Omega(u,v)$. Since $T_x(\alpha_x \cap \beta_x) \subset (T_{\alpha(x)}\mathscr{L}_{\alpha(x)})^\Omega$, we have $\Omega_B(u,v) = \Omega(u,v) = \Omega(u_2,v) \neq 0$.

Suppose $u_2 = 0$. Let $v \in T_{\alpha(x)}P/T_{\alpha(x)}\mathscr{L}_{\alpha(x)}$ be such that $\Omega(u,v) \neq 0$. Hence, $\Omega_B(u,v) = \Omega(u,v) \neq 0$.

Claim 4.4.3. We have $(T_x\alpha_x)^{\Omega_B} = (T_x\alpha_x)^\Omega$, $(T_x\beta_x)^{\Omega_B} = (T_x\beta_x)^\Omega$.

Let $v \in (T_x\alpha_x)^\Omega = T_x\beta_x$. For any $w \in T_x\alpha_x$ we have

$$\Omega_B(v,w) = \Omega(v,w) - \beta^* B(v,w) = 0 - B(T\beta(v), T\beta(w)) = 0.$$

Therefore, $(T_x\alpha_x)^\Omega = T_x\beta_x \subseteq (T_x\alpha_x)^{\Omega_B}$. In a similar way, $(T_x\beta_x)^\Omega = T_x\alpha_x \subseteq (T_x\beta_x)^{\Omega_B}$.

Let $v \in (T_x\alpha_x)^{\Omega_B}$. For any $w \in T_x\alpha_x$

$$0 = \Omega_B(v,w) = \Omega(v,w) - \beta^* B(v,w) = \Omega(v,w) - B(T\beta(v), T\beta(w)) = \Omega(v,w)$$

Therefore, $(T_x\alpha_x)^{\Omega_B} \subseteq (T_x\alpha_x)^\Omega$ and analogously $(T_x\beta_x)^\Omega \subseteq (T_x\beta_x)^{\Omega_B}$. □

□

Claim 4.4.4. The fibers of α and β are symplectically orthogonal with respect to Ω_B and $\{\alpha^* C^\infty(P_1), \beta^* C^\infty(P_2)\} = 0$.

Proof. From Claims 4.4.2 and 4.4.3 we obtain

$$T_x\beta_x = (T_x\alpha_x)^{\Omega_B} = \{X^B_{\alpha^* f} \,|\, f \in C^\infty(P)\},$$
$$T_x\alpha_x = (T_x\beta_x)^{\Omega_B} = \{X^B_{\beta^* f} \,|\, f \in C^\infty(P)\},$$

where X^B_g is the Hamiltonian vector field of $g \in C^\infty(S)$ with respect to the modified symplectic form Ω_B.

Claim 4.4.5. The map $\alpha : (S, \Omega_B) \to (P, \pi)$ is Poisson.

$\square$

Proof. The map α is Poisson iff $(T\alpha)X^B_{\alpha^* f} = X_f$ for all $f \in C^\infty(P)$, where $X^B_{\alpha^* f}$ is the Hamiltonian vector field of $\alpha^* f \in C^\infty(S)$ with respect to the symplectic form Ω_B. We have

$$d(\alpha^* f) = \widetilde{\Omega_B}(X^B_{\alpha^* f}) = \tilde{\Omega}(X^B_{\alpha^* f}) + \widetilde{\beta^* B}(X^B_{\alpha^* f}) = \tilde{\Omega}(X^B_{\alpha^* f}),$$

(where $\widetilde{\beta^* B}(X^B_{\alpha^* f}) = 0$ since $X^B_{\alpha^* f} \in T_x\beta_x$ by 4.4.4). Therefore, $X^B_{\alpha^* f} = X_{\alpha^* f}$. $\square$

Claim 4.4.6. The map $\beta : (S, \Omega_B) \to (P, \pi_B)$ is anti-Poisson.

Proof. The form Ω_B is obtained from Ω by the gauge transformation of (S, Ω) by the 2-form $-(T^*\beta)B$. Applying Theorem 4.3.1 to the map $\beta : (S, \Omega) \to (P, \pi)$, we obtain that $\beta : (S, \tau_{-(T^*\beta)B}\Omega) \to (P, \tau_B \pi)$ is an anti-Poisson map. $\square$

Claim 4.4.7. We have $X^B_{\alpha^* f} = X_{\alpha^* f}$, $X^B_{\beta^* f} = \hat{X}_{\beta^* f}$, where $\hat{X}_g$ is the Hamiltonian vector field of $g \in C^\infty(S)$ with respect to $\hat{\Omega}$.

Proof. Since $X_{\alpha^* f} \in \ker\beta$, we have $\beta^* B(X_{\alpha^* f}) = 0$. Hence, $\Omega_B(X_{\alpha^* f}) = \Omega(X_{\alpha^* f}) = d(\alpha^* f)$. Therefore, $X^B_{\alpha^* f} = X_{\alpha^* f}$. The other relation follows by symmetry.

Claim 4.4.8. The maps $\alpha : (S, \Omega_B) \to (P, \pi)$ and $\beta : (S, \Omega_B) \to (P, \pi_B)$ are complete.

Proof. Let $f \in C^\infty(P)$ be a complete function with respect to π. By Claim 4.4.7, $X^B_{\alpha^* f} = X_{\alpha^* f}$, which is a complete vector field since the source map $\alpha : (S, \Omega) \to (P, \pi)$ of a symplectic groupoid is complete (see, e.g., [Daz90], Sec. 6). Analogously, since $(S, \hat{\Omega})$ is a symplectic groupoid for (P, π_B), it follows that $\beta : (S, \hat{\Omega}) \to (P, \pi_B)$ is complete, and hence $X^B_{\beta^* f} = \hat{X}_{\beta^* f}$ is complete as well. $\square$

$\square$

Since two Morita equivalent Poisson structures on the same underlying manifold do not necessarily have the same leaf decomposition, it is easy to see that Morita equivalence does not imply gauge equivalence. Moreover, as was shown in [BR], even if we consider gauge equivalence up to a diffeomorphism, it is still not implied by Morita equivalence.

4.5 Example: topologically stable Poisson structures on a compact oriented surface

In this section we discuss gauge and Morita equivalence of Poisson structures with degeneracies of the first order on a compact connected oriented surface. We refer the reader to Chapter 3 for a discussion of these structures. In particular we recall (Theorem 3.2.13) that topologically stable Poisson structures $\mathcal{G}(\Sigma)$ can be classified up to a Poisson isomorphism by a finite number of invariants: the topology of the inclusion $Z(\pi) \subset \Sigma$ and $(n+1)$ numerical invariants: the n modular periods of π around each connected component of the zero set $Z(\pi)$ and the regularized Liouville volume (obtained as a certain regularized sum of symplectic volumes of two-dimensional leaves, taken with appropriate signs).

4.5.1 Gauge equivalence

An obvious necessary condition for two topologically stable structures π and π' on a surface to be gauge-equivalent is $Z(\pi) = Z(\pi')$, i.e. the zero sets of both structures should be the same. The following theorem gives the necessary and sufficient conditions for gauge-equivalence of two topologically stable structures:

Theorem 4.5.1. *Two topologically stable Poisson structures $\pi, \pi' \in \mathcal{G}_n(\Sigma)$ with the zero set $Z(\pi) = Z(\pi') = \bigsqcup_{i=1}^n \gamma_i$ are gauge equivalent if and only if their modular periods are the same around all the zero curves, i.e. $T_{\gamma_i}(\pi) = T_{\gamma_i}(\pi')$ for $i = 1, \ldots, n$.*

Proof. Modular periods are clearly an invariant of gauge equivalence.

Let $\pi = f \cdot \pi_0, \pi' = f' \cdot \pi_0$, where $f,\ f' \in C^\infty(\Sigma)$ are functions vanishing linearly on $Z(\pi) = Z(\pi') = \bigsqcup_{i=1}^n \gamma_i$ and non-zero elsewhere. Assume that $T_{\gamma_i}(\pi) = T_{\gamma_i}(\pi')$ for all $i = 1, \ldots, n$. We will explicitly find a 2-form $B \in \Omega^2(\Sigma)$ such that $\tilde{\pi}' = \tilde{\pi} \cdot (1 + \tilde{B} \circ \tilde{\pi})^{-1}$. First, define a 2-form $B|_{\Sigma \setminus Z(\pi)}$ on $\Sigma \setminus Z(\pi)$ by

$$B|_{\Sigma \setminus Z(\pi)} = \omega' - \omega = \left(\frac{1}{f'} - \frac{1}{f} \right) \omega_0 \tag{4.5.1}$$

The question is whether $B|_{\Sigma \setminus Z(\pi)}$ can be extended smoothly to a (closed) 2-form $B \in \Omega^2(\Sigma)$ on Σ.

For each $i = 1, \ldots, n$, let $U_i = \{(z_i, \theta_i) | \, |z_i| < R_i,\, \theta_i \in [0, 2\pi]\}$ be a small annular neighborhood of the zero curve $\gamma_i \in Z(\pi)$ such that $Z(\pi) \cap U_i = \gamma_i$ and $\pi|_{U_i} = c_i f(z_i) \partial_{z_i} \wedge \partial_{\theta_i}$, $\pi'|_{U_i} = f'(z_i) \partial_{z_i} \wedge \partial_{\theta_i}$ with $f|_{U_i} = c_i z_i + O(z^2)$, $f'|_{U_i} = c_i' z_i + O(z^2)$. One can then compute that $T_{\gamma_i}(\pi) = \frac{2\pi}{c}$,

$T_{\gamma_i}(\pi') = \frac{2\pi}{c'}$. Thus on $U_i \setminus \gamma_i$ we have

$$\frac{1}{f'} - \frac{1}{f} = \frac{f - f'}{ff'} = \frac{cz_i - c'z_i + O(z_i^2)}{z_i^2(1 + O(z_i))} = \frac{c - c'}{z_i} \cdot O(1) + O(1), \quad z_i \neq 0.$$

Therefore, $B|_{\Sigma \setminus Z(\pi)}$ can be extended from $U \setminus \gamma_i$ to U if and only if $c_i = c_i'$. Hence B can be defined as a smooth form on Σ if and only if $c_i = c_i'$ for all $i = 1, \dots, n$, i.e., the modular periods of π and π' around all curves are pairwise equal. Then $\pi' = \tau_B(\pi)$ and therefore the structures π and π' are gauge-equivalent. $\square$

Let $\mathscr{G}_n(\Sigma; \gamma_1, \dots, \gamma_n; T_{\gamma_1}, \dots, T_{\gamma_n})$ be the space of topologically stable Poisson structures $\pi \in \mathscr{G}_n(\Sigma)$ which have the same zero set $Z(\pi) = \bigsqcup_{i=1}^n \gamma_i$ and the same modular periods $T_{\gamma_1}, \dots, T_{\gamma_n}$ around the zero curves.

Corollary 4.5.2. *The additive group of closed 2-forms on Σ acts transitively on the space $\mathscr{G}_n(\Sigma; \gamma_1, \dots, \gamma_n; T_{\gamma_1}, \dots, T_{\gamma_n})$. The regularized Liouville volume changes under this action in the following way: $V(\tau_B\pi) = V(\pi) + Vol(B)$, where $Vol(B)$ is the Liouville volume of B.*

For $\pi \in \mathscr{G}_n(\Sigma)$, let $\mathscr{A}(\pi) = (T^*\Sigma, \rho, [\,,\,])$ be the Lie algebroid of the Poisson manifold (Σ, π). The anchor $\rho = -\tilde{\pi} : T^*\Sigma \to T\Sigma$ of this Lie algebroid is injective on the open dense set $\Sigma \setminus Z(\pi)$. According to a theorem of Debord [Deb00], a Lie algebroid with an almost injective anchor (i.e., injective on an open dense set) is integrable. Therefore, (Σ, π) is an integrable Poisson manifold.

Since for integrable Poisson manifolds gauge equivalence implies Morita equivalence, we have

Theorem 4.5.3. *Two topologically stable Poisson structures $\pi, \pi' \in \mathscr{G}_n(\Sigma)$ with the same zero set $Z(\pi) = Z(\pi') = \bigsqcup_{i=1}^n \gamma_i$ and equal modular periods, $T_{\gamma_i}(\pi) = T_{\gamma_i}(\pi')$ for $i = 1, \dots, n$ are Morita equivalent.*

4.5.2 Morita equivalence of topologically stable Poisson structures on S^2

In the case of two-sphere we will show that two Morita-equivalent topologically stable Poisson structures have topologically equivalent zero sets and their corresponding modular periods are equal.

Let M be the two-sphere. Let $\pi \in \mathscr{G}(M)$ be a topologically stable Poisson structure on M. Let as before $\pi = f \cdot \pi_0$, where π_0 is a non-degenerate Poisson structure on M and $f \in C^\infty(M)$ is a smooth function. The class of $Z(\pi)$ modulo diffeomorphisms of M is a Poisson isomorphism invariant. This class depends only on the topological arrangement of the curves comprising $Z(\pi)$ and can be described combinatorially by a signed tree $Tree(\pi)$ (see section 3.2.8).

Lemma 4.5.4. *If two topologically stable Poisson structures $\pi, \pi' \in \mathscr{G}_n(M)$ are Morita-equivalent, then there exists an isomorphism of trees $\phi : Tree(\pi) \to Tree(\pi')$, so that $T_\gamma(\pi) = T_{\phi(\gamma)}(\pi')$ for every edge γ (corresponding to $\gamma \in Z(\pi)$) of $Tree(\pi)$.*

Proof. Assume that (M,π) and (M,π') are Morita-equivalent. Denote as before by ϕ the isomorphism of the leaf spaces of (M,π) and (M,π'). By Proposition 4.2.1, ϕ is a homeomorphism of topological spaces.

As a set, the leaf space L of (M,π) can be identified with the union $Z(\pi) \sqcup \{\ell_1, \cdots, \ell_n\}$, where $\ell_1, \ldots, \ell_n$ are the points corresponding to the 2-dimensional leaves $\mathscr{L}_1, \ldots, \mathscr{L}_n$. The quotient topology of L is easily described: the only open subsets of L have the form $U \cup \{\ell_{i_1}\} \cup \cdots \cup \{\ell_{i_k}\}$, where $i_1, \ldots, i_k \in [1,n]$, $k \geq 0$ and $U \subset Z(\pi)$ is an open subset with the property that if U intersects non-trivially a curve $\gamma \subset Z(\pi)$, then for both leaves bounding γ the corresponding points of the leaf space occur among $\{\ell_{i_1}, \ldots, \ell_{i_k}\}$.

Now, given L with its topology, consider the collection $\mathscr{Y}$ of all subsets $Y \subset L$ with the property that $L \setminus Y$ is Hausdorff. Order $\mathscr{Y}$ by inclusion. We claim that $X = \{\ell_1, \ldots, \ell_n\}$ is a minimal element of $\mathscr{Y}$ of finite cardinality. First, note that $X \in \mathscr{Y}$, since the relative topology on $L \setminus X = Z(\pi) \subset L$ is Hausdorff. Next, assume that $Y \in \mathscr{Y}$, and $Y \not\supset X$. Then $\ell_i \in L \setminus Y$ for some $i \in [1,n]$. Now all of the points of the boundary of $\mathscr{L}_i$ in M lie in $Z(\pi)$ and cannot be separated from ℓ_i by open sets; thus all of these points must necessarily be in Y. Thus Y must have infinite cardinality.

It follows that ϕ must map X to a subset of L' with the same minimality property; and hence ϕ must take the complement of X, $Z(\pi)$, to $Z(\pi')$. Thus ϕ induces a map between the set of vertices of $Tree(\pi)$ and $Tree(\pi')$.

Now, two vertices $\ell_i, \ell_j \in Tree(\pi)$ are connected by an edge iff the corresponding regions share a boundary in M. A point $x \in Z(\pi) \subset L$ cannot be separated from ℓ_k by an open set if and only if x belongs to the boundary of $\mathscr{L}_k$ in M. It follows that ℓ_i, ℓ_j are connected by an edge iff there exists a point $x \in L$, such that $x \neq \ell_i$, $x \neq \ell_j$, but which cannot be separated from either of them by an open set. Since ϕ is a homeomorphism, it must preserve this property, and thus ϕ induces a map of trees from $Tree(\pi)$ to $Tree(\pi')$. The statement about modular periods now follows from Corollary 4.2.4. $\square$

Theorem 4.5.5. *For two topologically stable Poisson structures π and π' on the two-sphere to* *be Morita-equivalent, it is necessary and sufficient that there exists an isomorphism of trees* $\phi : Tree(\pi) \to Tree(\pi')$ *so that* $T_\gamma(\pi) = T_{\phi(\gamma)}(\pi')$ *for every edge* γ *of* $Tree(\pi)$.

Proof. The necessity follows from Lemma 4.5.4.

Assume now that the isomorphism $\phi : Tree(\pi) \to Tree(\pi')$ satisfying the conditions above exists. Let $\psi : S^2 \to S^2$ be an orientation-reversing diffeomorphism. By replacing π' with $\psi_*\pi'$ (which is obviously Poisson-isomorphic, and hence Morita-equivalent to, π') if necessary, we may assume that $\phi : Tree(\pi) \to Tree(\pi')$ is an isomorphism of *signed* trees.

Choose a function $g \in C^\infty(S^2)$ supported on the interior of one of the two-dimensional leaves. Let $\pi'' = \pi' + g\pi'$. Since $\pi'' = \pi'$ in a neighborhood of each of the zero curves $\gamma \subset Z(\pi)$, the modular periods of π' and π'' are equal. Therefore, by Theorem 4.5.1, π' and π'' are gauge-equivalent. Hence by Theorem 4.4.1, π' and π'' are Morita-equivalent, for any such choice of g. Also, the isomorphism ϕ induces an isomorphism of trees $\phi' : Tree(\pi) \to Tree(\pi'')$.

With a suitable choice of g, the regularized Liouville volume of π'' can be made equal to that of π (see §3.2.5 for details). Thus by Theorem 3.2.13, π and π'' are Poisson-isomorphic. We conclude that π and π'' are Morita-equivalent, since they are integrable (see [Deb00]; note that the structures involved are symplectic except on a dense set). Thus π and π' are also Morita-equivalent, by transitivity of Morita equivalence. $\square$

Chapter 5

Poisson cohomology of the r-matrix Poisson structure on $SU(2)$.

5.1 Poisson-Lie groups as examples of Poisson manifolds.

A Poisson-Lie group is a group object in the Poisson category, where objects are Poisson manifolds and maps are Poisson maps.

The notion of a Poisson-Lie group was introduced and the theory of these objects was developed in the works of Drinfeld [Dri83] and Semenov-Tian-Shansky [STS83], [STS85], motivated by attempts to describe the Hamiltonian structures of the groups of dressing transformations of some integrable systems. Poisson-Lie groups are also the objects corresponding to the so-called quantum groups (in the sense of Drinfel'd [Dri87]) in the classical limit.

A Lie group G with a Poisson structure π is called a *Poisson-Lie group* if π satisfies the *multiplicativity condition*, i.e., the group multiplication $m : G\times G \to G$ is a Poisson map (where $G\times G$ is endowed with the product Poisson structure). In terms of Poisson bivector, the multiplicativity condition is equivalent to

$$\pi(gh) = \ell_g\pi(h) + r_h\pi(g), \qquad \forall g,h \in G, \tag{5.1.1}$$

where ℓ_g and r_h denote the extensions of the differentials of left and right translations on G by g and h to bivectors. In particular, (5.1.1) implies that $\pi(e) = 0$, and that the linearization of π at the identity element $e \in G$ gives rise to a well-defined map $\delta : \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$, called the *cobracket* on the

Lie algebra $\mathfrak{g}$ of the Lie group G:

$$\delta(X) = \frac{d}{dt}|_{t=0}\pi(\exp(tX))\exp(-tX). \tag{5.1.2}$$

The Jacobi identity for π implies that the adjoint map $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ also satisfies the Jacobi identity and therefore defines a Lie bracket on $\mathfrak{g}^*$ (thus the name "cobracket" for the map δ).

The multiplicativity of π implies the following cocycle property for $\delta \in \mathfrak{g}^* \otimes \Lambda^2\mathfrak{g}$:

$$\delta([X,Y]) = [X,\delta(Y)] - [Y,\delta(X)]. \tag{5.1.3}$$

This property can be interpreted as a compatibility condition for the Lie brackets $[\cdot,\cdot]$ on $\mathfrak{g}$ and δ^* on $\mathfrak{g}^*$: these brackets can be extended to a Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ if and only if the condition (5.1.3) is satisfied. A Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ together with a map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ satisfying (5.1.3) is called a Lie bialgebra. The *Lie bialgebra* $(\mathfrak{g},[\cdot,\cdot],\delta)$ with δ defined by (5.1.2) is called the Lie bialgebra of the Poisson-Lie group (G,π). If the Lie group G is connected and simply connected, then (G,π) is completely determined by the Lie bialgebra $(\mathfrak{g},[\cdot,\cdot],\delta)$: the Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ determines G as a Lie group, while the comultiplication δ can be used to recover the Poisson structure π.

It is particularly easy to recover the Poisson structure on G under the additional assumption that $\delta \in \mathfrak{g}^* \otimes \Lambda^2\mathfrak{g}$ is a coboundary, i.e., that there exists an element $r \in \Lambda^2\mathfrak{g}$, so that

$$\delta(X) = -[X,r], \quad \forall X \in \mathfrak{g}. \tag{5.1.4}$$

In this case, the Jacobi identity for δ^* is equivalent to the *modified classical Yang-Baxter equation*, which states that the element $[\![r,r]\!] \in \Lambda^3\mathfrak{g}$ be ad-invariant (here $[\![\cdot,\cdot]\!]$ denotes the algebraic Schouten bracket on $\Lambda^*\mathfrak{g}$). An element $r \in \Lambda^2\mathfrak{g}$ satisfying the modified classical Yang-Baxter equation is called a *classical r-matrix.* If the cobracket δ on $\mathfrak{g}$ is given by such an r-matrix as explained above, the corresponding Poisson structure on G is given by the simple equation

$$\pi(g) = \ell_g(r) - r_g(r), \qquad \forall g \in G, \tag{5.1.5}$$

where ℓ_g and r_g denote, as before, the extensions of the differentials of left and right translations by g to $\Lambda^2\mathfrak{g}$.

5.2 The standard r-matrix Poisson structure on $SU(2)$.

Consider the Lie group $SU(2)$ identified with the unit sphere in $\mathbb{C}^2$ as follows

$$SU(2) = \left\{ A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z,w \in \mathbb{C}, \det A = z\bar{z} + w\bar{w} = 1 \right\}.$$

Let $su(2)$ be the corresponding Lie algebra with the basis $\{e_1, e_2, e_3\}$ given by

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie bracket is determined by $[e_i, e_j] = \varepsilon_{ijk} e_k$, where ε_{ijk} is the completely skew-symmetric symbol. The standard r-matrix $r = e_1 \wedge e_2 \in \Lambda^2 su(2)$, satisfying the modified Yang-Baxter equation, defines a multiplicative Poisson structure on $SU(2)$ by

$$\pi_{SU(2)}(A) = rA - Ar.$$

The Poisson brackets between the coordinate functions $(z, \bar{z}, w, \bar{w})$ on $SU(2)$ are given by

$$\{z, \bar{z}\} = -i|w|^2, \quad \{w, \bar{w}\} = 0, \quad \{z, w\} = \frac{1}{2} izw, \quad \{z, \bar{w}\} = \frac{1}{2} z\bar{w}.$$

The symplectic leaves of this structure are the two-dimensional discs

$$D_\theta = \left\{ A = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in SU(2) : \arg w = \theta \right\}, \quad \theta \in [0, 2\pi)$$

and the points of the circle $N = \{A \in SU(2) : w = \bar{w} = 0\}$ which bounds each of the disks D_θ.

To compute the Poisson cohomology of this Poisson structure we cover $SU(2)$ with two open sets U and V described below, then compute the cohomologies of U, V and $U \cap V$ and use the Mayer-Vietoris exact sequence for Poisson cohomology.

Let

$$U \doteq \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in SU(2) : |z|^2 > 1/3 \right\}, \tag{5.2.1}$$

$$V \doteq \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in SU(2) : |w|^2 > 1/3 \right\}. \tag{5.2.2}$$

Since $|z|^2 + |w|^2 = 1$, it can not happen that both $|z|^2$ and $|w|^2$ are less than $1/3$. Therefore, U and V cover all of $SU(2)$.

It is useful to have an explicit description of the restriction of the Poisson structure $\pi_{SU(2)}$ to U and V. Consider the disc $C = \{w \in \mathbb{C} : |w|^2 < 2/3\}$ and the unit circle $S = \{s \in \mathbb{C} : |s| = 1\}$. Then the map

$$\Psi : C \times S \ni (w, s) \mapsto \begin{pmatrix} (1-|w|^2)^{1/2} s & -\bar{w} \\ w & (1-|w|^2)^{1/2} \bar{s} \end{pmatrix} \in U$$

is a diffeomorphism of $C \times S$ onto U. The restriction of the Poisson structure to U identified with $C \times S$ is then given by

$$\{s,\bar{s}\} = 0, \quad \{w,\bar{w}\} = 0, \quad \{s,w\} = \frac{1}{2}isw, \quad \{s,\bar{w}\} = \frac{1}{2}is\bar{w}.$$

If we write $w = x + iy$, $\bar{w} = x - iy$, and $s = e^{i\phi}$, the Poisson tensor becomes $\pi_U = -(x\partial_x + y\partial_y) \wedge \partial_\phi$. It will be also sometimes convenient to use the polar coordinates (r,θ) on C, given by $r = |w|$, $\theta = \arg w$. In coordinates (r,θ,ϕ) we have $\pi_U = r\partial_r \wedge \partial_\phi$.

Consider now the disc $D = \{z \in \mathbb{C} : |z|^2 < 2/3\}$ and the unit circle $T = \{t \in \mathbb{C} : |t| = 1\}$. The map

$$\Phi : D \times T \ni (z,t) \mapsto \begin{pmatrix} z & -(1-|z|^2)^{1/2}\bar{t} \\ (1-|z|^2)^{1/2}t & \bar{z} \end{pmatrix} \in V$$

is a diffeomorphism of $D \times T$ onto V. The pull-back of the Poisson structure on V to $D \times T$ is given by the formulas

$$\{z,\bar{z}\} = -i(1-|z|^2)^{1/2}, \quad \{z,t\} = \{\bar{z},t\} = 0.$$

It follows that the Poisson structure on $D \times T$ is the product of a symplectic structure on D (with infinite total volume) and the trivial Poisson structure on T.

The intersection $U \cap V$ is isomorphic to

$$\Phi^{-1}(U \cap V) = \{(z,t) \in D \times T : 1/3 < |z|^2 < 2/3\};$$

This manifold is manifestly Poisson-isomorphic to the product of the circle T (with the zero Poisson structure) and the annulus $A = \{z \in \mathbb{C} : 1/3 < |z|^2 < 2/3\}$ with a finite-volume symplectic structure.

5.3 Cohomology of U.

Recall that, as a Poisson manifold, U has the following description. As a manifold, U is isomorphic to the solid 2-torus $C \times S$, where C is a disc and S is a circle and the Poisson tensor on U is given by

$$\pi_U = (x\partial_x + y\partial_y) \wedge \partial_\phi, \tag{5.3.1}$$

where (x,y) are the usual coordinates on C resulting from its identification with a disc on a two-dimensional plane, and ϕ is a periodic coordinate on S. The symplectic leaves of π_U are the points of the circle $\{(x,y,\phi) : x = y = 0\}$ and the annuli $\{(x,y,\phi) : x/y = \text{const}, x,y \text{ not both zero}\}$. In this section we will compute the Poisson cohomology of (U,π_U).

By an argument similar to the one in [Gin96], the complexified Poisson cohomology of U is isomorphic to the cohomology of the complex

$$\mathbb{X}^0 \to \mathbb{X}^1 \to \mathbb{X}^2 \to \mathbb{X}^3,$$

where $\mathbb{X}^k$ is the space of k-vector fields on U, whose coefficients (as functions of x, y and ϕ) are formal power series in x, y and Fourier series in $e^{i\phi}$. Let

$$f = \sum_{n,m\geq 0, p\in\mathbb{Z}} a_{n,m,p} x^n y^m e^{ip\phi} \in \mathbb{X}^0, \tag{5.3.2}$$

$$X = \sum_{n,m\geq 0, p\in\mathbb{Z}} x^n y^m e^{ip\phi} \left(b^x_{n,m,p}\partial_x + b^y_{n,m,p}\partial_y + b^\phi_{n,m,p}\partial_\phi \right) \in \mathbb{X}^1, \tag{5.3.3}$$

$$Y = \sum_{n,m\geq 0, p\in\mathbb{Z}} x^n y^m e^{ip\phi} \left(c^x_{n,m,p}\partial_y\wedge\partial_\phi + c^y_{n,m,p}\partial_x\wedge\partial_\phi + c^\phi_{n,m,p}\partial_x\wedge\partial_y \right) \in \mathbb{X}^2, \tag{5.3.4}$$

where $a_{n,m,p}$, $b^x_{n,m,p}$, $b^y_{n,m,p}$, $b^\phi_{n,m,p}$, $c^x_{n,m,p,}$ $c^y_{n,m,p}$, $c^\phi_{n,m,p} \in \mathbb{C}$. The notation for f, X, Y will be fixed for the remainder of the section. A direct computation shows that we have:

$$\begin{aligned} d_\pi f &= \sum a_{n,m,p}(n+m)x^n y^m e^{ip\phi}\partial_\phi \\ &\quad - \sum ipa_{n,m,p}x^{n+1}y^m e^{ip\phi}\partial_x \\ &\quad - \sum ipa_{n,m,p}x^n y^{m+1} e^{ip\phi}\partial_y; \end{aligned}$$

$$\begin{aligned} d_\pi X &= \sum \left[ipb^x_{n,m,p}x^n y^{m+1}e^{ip\phi} - ipb^y_{n,m,p}x^{n+1}y^m e^{ip\phi} \right] \partial_x\wedge\partial_y \\ &\quad + \sum \left[-b^x_{n,m,p}(-1+n+m)x^n y^m e^{ip\phi} - ipb^\phi_{n,m,p}x^{n+1}y^m e^{ip\phi} \right] \partial_x\wedge\partial_\phi \\ &\quad + \sum \left[-b^y_{n,m,p}(-1+n+m)x^n y^m e^{ip\phi} - ipb^\phi_{n,m,p}x^n y^{m+1} e^{ip\phi} \right] \partial_y\wedge\partial_\phi; \end{aligned}$$

$$\begin{aligned} d_\pi Y &= \partial_x\wedge\partial_y\wedge\partial_\phi \times \\ &\times \Big(\sum (n+m-2)c^\phi_{n,m,p}x^n y^m e^{ip\phi} \\ &\quad - \sum ipc^x_{n,m,p}x^{n+1}y^m e^{ip\phi} \\ &\quad + \sum ipc^y_{n,m,p}x^n y^{m+1}e^{ip\phi} \Big). \end{aligned}$$

We now turn to the question of computing the Poisson cohomology.

5.3.1 The zeroth cohomology.

The zeroth Poisson cohomology is generated by the functions constant on all of the symplectic leaves of U. Since the circle $\{(x,y,\phi) : x = y = 0\}$ is in the closure of every two-dimensional leaf and since the two-dimensional leaves together form a dense subset of U, such a function must be constant. Thus $H^0_\pi(U) \simeq \mathbb{C} = \mathrm{span}\langle 1 \rangle$.

5.3.2 The first cohomology.

Let $X \in \mathbb{X}^1$, $d_\pi X = 0$. This is equivalent to the following conditions:

1. The coefficient of $\partial_x \wedge \partial_y$ is zero in the expression for $d_\pi X$. This implies
   (a) $pb^x_{1+n,m,p} = pb^y_{n,1+m,p}$ for all n,m,p; and
   (b) $pb^x_{0,m,p} = 0$ and $pb^y_{n,0,p} = 0$ for all n,m,p;
2. The coefficient of $\partial_x \wedge \partial_\phi$ is zero. This implies
   (a) $(m-1)b^x_{0,m,p} = 0$ for all m and p; and
   (b) $ipb^\phi_{n,m,p} = -(n+m)b^x_{n+1,m,p}$ for all n,m,p;
3. The coefficient of $\partial_y \wedge \partial_\phi$ is zero. This means that:
   (a) $(n-1)b^y_{n,0,p} = 0$ for all n and p; and
   (b) $ipb^\phi_{n,m,p} = -(n+m)b^y_{n,m+1,p}$;

In order to compute the first cohomology, we have to find out when $d_\pi X = 0$ for $X \in X^1$ implies that $X = d_\pi f$ for some $f \in X^0$. We claim that $X = d_\pi f$ if and only if the following conditions are satisfied:

$$b^x_{1,0,0} = b^y_{0,1,0} = b^x_{0,1,0} = b^y_{1,0,0} = b^\phi_{0,0,0} = 0. \tag{5.3.5}$$

Indeed, if $X = d_\pi f$ with f as in (5.3.2), we find that the only terms with $p = 0$ in the expression for $d_\pi f$ are

$$\sum (n+m)a_{n,m,0}x^n y^m \partial_\phi.$$

Thus necessarily $b^x_{n,m,0} = b^y_{n,m,0} = 0$ for all n and m. Also, $b^\phi_{0,0,0} = (0+0)a_{0,0,0} = 0$. So the conditions (5.3.5) must be satisfied.

Conversely, if the conditions (5.3.5) are satisfied, let f be given by (5.3.2) where we set for $p \neq 0$

$$\begin{aligned} a_{n,m,p} &= \frac{1}{-ip} b^x_{n+1,m,p}, \quad p \neq 0, \\ a_{n,m,0} &= \frac{1}{n+m} b^\phi_{n,m,0}, \quad n+m \neq 0, \\ a_{0,0,0} &= 0. \end{aligned}$$

In this case, $d_\pi f$ is given by

$$\begin{aligned} d_\pi f &= \sum_{p\neq 0} \frac{1}{-ip} b^x_{n+1,m,p}(n+m)x^n y^m e^{ip\phi}\partial_\phi \\ &\quad + \sum_{n+m\neq 0} b^\phi_{n,m,0} x^n y^m e^{i0\phi}\partial_\phi \\ &\quad + + \sum_{p\neq 0} b^x_{n+1,m,p} x^{n+1} y^m e^{ip\phi}\partial_x \\ &\quad \sum_{p\neq 0} b^x_{n+1,m,p} x^n y^{m+1} e^{ip\phi}\partial_y. \end{aligned}$$

We now claim that with these definitions $X = d_\pi f$. First, by condition (2a), the coefficients of ∂_ϕ in $d_\pi f$ and X are the same when $p \neq 0$. By (1b), the coefficient of $y^m e^{ip\phi}\partial_x$ in the expression for X is zero, so the coefficients of ∂_x in X and $d_\pi f$ are the same, when $p \neq 0$. Finally, by (1a) we conclude that $b^x_{n+1,m,p} = b^y_{n,m+1,p}$. Using (1b) again, we get that the coefficients of ∂_y in the expressions for X and $d_\pi f$ are the same when $p \neq 0$, so that the coefficients of $e^{ip\phi}$ for $p \neq 0$ are the same in X and $d_\pi f$.

For the terms with $p = 0$, because of the assumptions (5.3.5) we get that the coefficients of ∂_ϕ are the same for both $d_\pi f$ and X. It remains to prove that the all other coefficients in the expression for X must be zero; i.e., $b^x_{n,m,0} = b^y_{n,m,0} = 0$ for all n and m. The assumptions (5.3.5) give us this for n and m such that $n+m = 1$. If $n+m \neq 1$, we get the conclusion for b^x from (2a) for $n = 0$ and (2b) if $n \neq 0$; the conclusion for b^y follows from (3a) and (3b) in a similar way.

We, therefore, conclude that $H^1_\pi(U) \simeq \mathbb{C}^5 = \mathrm{span}\langle x\partial_x, y\partial_x, x\partial_y, y\partial_y, \partial_\phi\rangle$.

It is useful to have an expression for the first four generators in the polar coordinates (r,θ) on C:

$$\begin{aligned} x\partial_x &= r\cos^2\theta\partial_r - \cos\theta\sin\theta\partial_\theta, \\ y\partial_x &= r\cos\theta\sin\theta\partial_r - \sin^2\theta\partial_\theta, \\ x\partial_y &= r\cos\theta\sin\theta\partial_r + \cos^2\theta\partial_\theta, \\ y\partial_y &= r\sin^2\theta\partial_r + \cos\theta\sin\theta\partial_\theta. \end{aligned}$$

5.3.3 The second cohomology.

Assume that $Y \in \mathbb{X}^2$ is such that $d_\pi Y = 0$. This happens iff all of the following conditions are satisfied:

1. $c^\phi_{0,0,p} = 0$ for all p (this is obtained by considering the coefficient of $e^{ip\phi}\partial_x \wedge \partial_y \wedge \partial_\phi$);

2. $(m-1)c^{\phi}_{0,m+1,p} + ipc^{y}_{0,m,p} = 0$ for all p,m (obtained by considering the coefficient of $y^{m+1}e^{ip\phi}\partial_x \wedge \partial_y \wedge \partial_\phi$);

3. $(n-1)c^{\phi}_{n+1,0,p} - ipc^{x}_{n,0,p} = 0$ for all p and n (obtained by considering the coefficient of $x^{n+1}e^{ip\phi}\partial_x \wedge \partial_y \wedge \partial_\phi$);

4. $(n+m)c^{\phi}_{n+1,m+1,p} - ipc^{x}_{n,m+1,p} + ipc^{y}_{n+1,m,p} = 0$ (obtained by considering the coefficient of $x^{n+1}y^{m+1}e^{ip\phi}\partial_x \wedge \partial_y \wedge \partial_\phi$).

We claim that $Y = d_\pi X$ if and only if

$$c^{\phi}_{0,2,0} = c^{\phi}_{1,1,0} = c^{\phi}_{2,0,0} = c^{y}_{0,1,0} = c^{y}_{1,0,0} = c^{x}_{0,1,0} = c^{x}_{1,0,0} = 0. \tag{5.3.6}$$

Indeed, if $Y = d_\pi X$, we get from the expression for $d_\pi X$ that among the terms with $p = 0$

1. all coefficients of $\partial_x \wedge \partial_y$ must vanish (i.e., $c^{\phi}_{m,n,0} = 0$ for all m,n); and

2. all terms of the form $x^m y^n e^{i0\phi}\partial_x \wedge \partial_\phi$ and $x^m y^n e^{i0\phi}\partial_y \wedge \partial_\phi$ must vanish if $n+m = 1$.

Thus (5.3.6) is necessary for $Y = d_\pi X$.

Conversely, given Y satisfying the conditions (5.3.6), consider X whose coefficients are defined as follows:

$$\begin{aligned} b^{x}_{0,m,p} &= \frac{1}{ip}c^{\phi}_{0,m+1,p}, \quad p \neq 0, \\ b^{x}_{0,m,0} &= \frac{-1}{m-1}c^{y}_{0,m,0}, \quad m \neq 1, \\ b^{x}_{0,1,0} &= 0; \end{aligned}$$

$$\begin{aligned} b^{y}_{n,0,p} &= \frac{-1}{ip}c^{\phi}_{n+1,0,p}, \quad p \neq 0, \\ b^{y}_{n,0,0} &= \frac{-1}{n-1}c^{x}_{n,0,0}, \quad n \neq 1, \\ b^{y}_{0,1,0} &= 0. \end{aligned}$$

$$\begin{aligned} b^{x}_{1,0,0} &= 0, \\ b^{y}_{0,1,0} &= 0, \\ b^{\phi}_{n,m,0} &= 0. \end{aligned}$$

$$\begin{aligned}
b^x_{n+1,m,p} &= \frac{-1}{n+m}c^y_{n+1,m,p}, \quad n+m \neq 0,\\
b^y_{n,m+1,p} &= \frac{-1}{n+m}c^x_{n,m+1,p}, \quad n+m \neq 0,\\
b^\phi_{0,0,p} &= -\frac{1}{ip}c^y_{1,0,p}, \quad p \neq 0,\\
b^\phi_{n,m,p} &= 0, \quad n+m \neq 0, p \neq 0.
\end{aligned}$$

We now claim that $Y = d_\pi X$.

Consider in the expression for $d_\pi X$ the coefficient of $\partial_x \wedge \partial_y$. When $p = 0$ this coefficient is zero, and equals to the corresponding coefficient of Y (using (1) when $n = m = 0$; the assumptions (5.3.6) together with (2) when $n = 0$, $m \neq 0$, together with (3) when $m = 0$, $n \neq 0$ and together with (4) when $m,n \neq 0$). Assume now that $p \neq 0$. When $n = 0$ and $m = 0$, we get zero in the expression for $d_\pi X$ and also zero for $c^\phi_{0,0,p}$ because of (1). When $n = 0$, $m \neq 0$ we get $c^\phi_{0,m,p}x^n y^m e^{ip\phi}$ in the expression for $d_\pi X$, and similarly when $m = 0, n \neq 0$. When both n and m are nonzero, we get

$$\frac{-ip}{n+m}(c^y_{n,m,p} - c^x_{n,m,p}) = c^\phi_{n,m,p}$$

by (4). Thus the coefficients of $\partial_x \wedge \partial_y$ are the same in both the expression for $d_\pi X$ and Y.

Consider next the coefficient of $\partial_x \wedge \partial_\phi$. When $p = 0$, the corresponding coefficient in $d_\pi X$ is given by

$$-(-1+n+m)b^x_{n,m,0}x^n y^m e^{i0\phi}.$$

In the case that $n = 0$ and $m \neq 1$, using (2) we get exactly $c^y_{0,m,p}$; when $m = 1$ we get zero, just as in (5.3.6). In the case $m = 0$, $n \neq 1$ we get exactly $c^y_{0,m,p}$ by the definition of $b^x_{n,m,p}$; when $n = 1$ we get zero, just as in (5.3.6).

When $p \neq 0$ the coefficient of $e^{ip\phi}\partial_x \wedge \partial_\varphi$ is

$$\sum -b^x_{n,m,p}(-1+n+m)x^n y^m - ipb^\phi_{n,m,p}x^{n+1}y^m.$$

Looking first at the coefficient of $x^0 y^m$ in this expression, we get

$$-(m-1)\frac{1}{ip}c^\phi_{0,m+1,p}$$

as the coefficient coming from $d_\pi X$; using (2) we get exactly $c^y_{0,m,p}$. Looking now at the coefficient of $x^{n+1}y^m$ we get

$$\begin{aligned}
&-\frac{-1}{n+m}c^y_{n+1,m,p}(n+m), \qquad \text{if } n+m \neq 0\\
&-ipb^\phi_{0,0,p} = c^y_{1,0,p}, \qquad \text{if } n = m = 0
\end{aligned}$$

as desired.

We now look at the coefficient of $\partial_y \wedge \partial_\varphi$. We note that the definition of $b^*_{*,*,*}$ is symmetric with respect to changing the roles of x and y, with the only exception being the definition of $b^\phi_{0,0,p} = \frac{-1}{ip} c^y_{1,0,p}$. Using (4) we get that also $b^\phi_{0,0,p} = \frac{-1}{ip} c^x_{0,1,p}$; the rest now follows by symmetry. Thus $Y = d_\pi X$.

It follows that $H^2_\pi(U) \simeq \mathbb{C}^7$ and is spanned by the cohomology classes of the bivector fields $y^2\partial_x \wedge \partial_y$, $xy\partial_x \wedge \partial_y$, $x^2\partial_x \wedge \partial_y$, $x\partial_y \wedge \partial_\phi$, $y\partial_y \wedge \partial_\phi$, $x\partial_x \wedge \partial_\phi$ and $y\partial_x \wedge \partial_\phi$.

5.3.4 The third cohomology.

Let now $Z = \sum d_{n,m,p} x^n y^m e^{ip\phi} \partial_x \wedge \partial_y \wedge \partial_z$. For dimension reasons, we have $d_\pi Z = 0$. We now claim that $Z = d_\pi Y$ if and only if

$$d_{2,0,0} = d_{1,1,0} = d_{0,2,0} = 0. \tag{5.3.7}$$

Indeed, if $Z = d_\pi Y$, then the coefficients of x^2, xy and y^2 in the expression for Z must vanish (this corresponds to $n + m = 2$ and $p = 0$ in the formula for $d_\pi Y$), which implies (5.3.7).

Conversely, assume that (5.3.7) is satisfied. Then set

$$\begin{aligned} c^\phi_{n,m,0} &= \frac{1}{n+m-2} d_{n,m,0}, \quad n+m \neq 2 \\ c^\phi_{2,0,0} &= c^\phi_{1,1,0} = c^\phi_{0,2,0} = 0 \end{aligned}$$

Set also $c^x_{n,m,0} = c^y_{n,m,0} = 0$ for all n, m. For $p \neq 0$ let

$$\begin{aligned} c^\phi_{n,m,p} &= \frac{1}{n+m-2} d_{n,m,p}, \quad n+m \neq 2, \\ c^x_{n-1,m,p} &= 0, \quad n+m \neq 2, \\ c^y_{n,m-1,p} &= 0, \quad n+m \neq 2. \end{aligned}$$

In the remaining cases ($p \neq 0$ and $n + m = 2$, so that (n, m) is one of $(1, 1)$, $(0, 2)$, $(2, 0)$) set

$$\begin{aligned} c^\phi_{n,m,p} &= 0, \quad n+m = 2 \\ c^x_{1,0,p} &= \frac{-1}{ip} d_{2,0,p}, \\ c^y_{0,1,p} &= \frac{1}{ip} d_{0,2,p}, \\ c^x_{0,0,p} &= -c^y_{0,0,p} = \frac{-1}{2ip} d_{1,1,p}. \end{aligned}$$

We now check that $Z = d_\pi Y$. In the case that $p \neq 0$, we get in the expression for $d_\pi X$ the term

$$\left(\sum (n+m-2)c^{\phi}_{n,m,p}x^n y^m e^{ip\phi} - ipc^{x}_{n,m,p}x^{n+1}y^m e^{ip\phi} + ipc^{y}_{n,m,p}x^n y^{m+1}e^{ip\phi}\right)\partial_x \wedge \partial_y \wedge \partial_\phi.$$

Considering the coefficient of $x^n y^m$ in the case $n+m \neq 2$ we get $(n+m-2)d^{\phi}_{n,m,p}\frac{1}{(n+m-2)} = d^{\phi}_{n,m,p}$. In the case that $n+m=2$, we get

$$-ipc^{x}_{n-1,m,p} + ipc^{y}_{n,m-1,p} = d_{n,m,p}$$

by definition.

If $p = 0$, we obtain

$$\sum (n+m-2)c^{\phi}_{n,m,0}x^n y^m e^{i0\phi} = \sum_{n+m\neq 2} d_{n,m,p}x^n y^m e^{i0\phi} = Z$$

because of conditions (5.3.7).

We conclude that $H^3_\pi(U) \simeq \mathbb{C}^3$ and is spanned by the cohomology classes of the 3-vector fields $x^2\partial_x \wedge \partial_y \wedge \partial_\varphi$, $xy\partial_x \wedge \partial_y \wedge \partial_\varphi$ and $y^2\partial_x \wedge \partial_y \wedge \partial_\varphi$.

We summarize the results of this section:

Proposition 5.3.1. *The Poisson cohomology of (U,π_U) is given by:*

$$\begin{aligned}
H^0_\pi(U) &\simeq span\langle 1\rangle = \mathbb{C}^1\\
H^1_\pi(U) &\simeq span\langle x\partial_x, y\partial_x, x\partial_y, y\partial_y, \partial_\phi\rangle = \mathbb{C}^5\\
H^2_\pi(U) &\simeq span\langle y^2\partial_x \wedge \partial_y, xy\partial_x \wedge \partial_y, x^2\partial_x \wedge \partial_y, x\partial_y \wedge \partial_\phi, y\partial_y \wedge \partial_\phi, x\partial_x \wedge \partial_\phi, y\partial_x \wedge \partial_\phi\rangle = \mathbb{C}^7\\
H^3_\pi(U) &\simeq span\langle x^2\partial_x \wedge \partial_y \wedge \partial_\phi, xy\partial_x \wedge \partial_y \wedge \partial_\phi, y^2\partial_x \wedge \partial_y \wedge \partial_\phi\rangle = \mathbb{C}^3
\end{aligned}$$

5.4 The cohomologies of V and of $U \cap V$.

Recall (see, e.g., [Vai94, Corollary 5.14]) that the Poisson cohomology of a product Poisson manifold $P = M \times N$, where M is a symplectic manifold and N is a manifold with the zero structure and finite Betti numbers, is given by the Künneth formula

$$H^k_\pi(P) = \bigoplus_{l=0}^{k} H^l(M) \otimes \mathfrak{X}^{k-l}(N). \tag{5.4.1}$$

We will apply this result to find the cohomology of the neighborhoods U and $U \cap V$ introduced above. As a Poisson manifold, $V \cong D \times T$, where D is a symplectic disc of finite symplectic volume and T is a circle with the zero Poisson structure. Therefore

$$\begin{aligned} H^0_\pi(V) &= C^\infty(\theta), \\ H^1_\pi(V) &= \mathfrak{X}^1(\theta), \\ H^2_\pi(V) &= 0, \\ H^3_\pi(V) &= 0. \end{aligned}$$

where we write θ for the coordinate on the circle T, and $C^\infty(\theta)$ and $\mathfrak{X}^1(\theta)$ are the spaces of smooth functions and smooth vector fields on T respectively.

Similarly, $U \cap V \cong A \times T$ as a Poisson manifold, where A is an annulus of infinite symplectic volume. It follows that the Poisson cohomology of $U \cap V$ is given by (we use coordinates (r, θ, ϕ) introduced above)

$$\begin{aligned} H^0_\pi(U \cap V) &= C^\infty(\theta), \\ H^1_\pi(U \cap V) &= C^\infty(\theta) \cdot \mathrm{span}\langle r\partial_r \rangle \oplus \mathfrak{X}^1_\pi(\theta), \\ H^2_\pi(U \cap V) &= X^1_\pi(\theta) \wedge \mathrm{span}\langle r\partial_r \rangle, \\ H^3_\pi(U \cap V) &= 0. \end{aligned}$$

5.5 The Poisson cohomology of $SU(2)$

Let $M = SU(2) = U \cup V$. We have the following Mayer-Vietoris exact sequence for computing the Poisson cohomology

$$\begin{aligned} 0 \to H^0_\pi(M) \to H^0_\pi(U) \oplus H^0_\pi(V) &\xrightarrow{j_0} H^0_\pi(U \cap V) \xrightarrow{\delta_0} \\ \to H^1_\pi(M) \xrightarrow{i_1} H^1_\pi(U) \oplus H^1_\pi(V) &\xrightarrow{j_1} H^1_\pi(U \cap V) \xrightarrow{\delta_1} \\ \to H^2_\pi(M) \xrightarrow{i_2} H^2_\pi(U) \oplus H^2_\pi(V) &\xrightarrow{j_2} H^2_\pi(U \cap V) \xrightarrow{\delta_2} \\ \to H^3_\pi(M) \xrightarrow{i_3} H^3_\pi(U) \oplus H^3_\pi(V) &\to H^3_\pi(U \cap V) \to \quad 0. \end{aligned}$$

5.5.1 The zeroth cohomology.

Since the set of two-dimensional leaves of $\pi_{SU(2)}$ is dense in $SU(2)$ and all these leaves have a common circle in their closures, it follows that the only Casimir functions on $SU(2)$ are constants. Thus, $H^0_\pi(SU(2)) \simeq \mathbb{C}^1$.

5.5.2 The first cohomology.

From the first and second rows of the Mayer-Vietoris sequence, we have $H^1_\pi(M) \simeq \mathrm{Im}\delta_0 \oplus \mathrm{Im} i_1$. Since the map j_0 is onto, by exactness it follows that $\mathrm{Im}\delta_0 = 0$. The map i_1 is injective, and $\mathrm{Im} i_1 \simeq \ker j_1$. It is easy to see that $([\partial_\phi], 0) \in \ker j_1$. For the remaining generators of $H^1_\pi(U)$, let $u = ax\partial_x + by\partial_x + cx\partial_y + dy\partial_y$, where $a, b, c, d \in \mathbb{C}$. We have

$$j_1([u]_U, [f(\theta)\partial_\theta]_V) = \tag{5.5.1}$$

$$[(-f(\theta) - a\cos\theta\sin\theta - b\sin^2\theta + c\cos^2\theta + d\cos\theta\sin\theta)\cdot\partial_\theta]_{U\cap V} + \tag{5.5.2}$$

$$[(a\cos^2\theta + b\cos\theta\sin\theta + c\cos\theta\sin\theta + d\sin^2\theta)\cdot r\partial_r]_{U\cap V}. \tag{5.5.3}$$

Therefore, $j_1([u]_U, [f(\theta)\partial_\theta]_V) = 0$ iff $a = d = 0$, $b = -c$ and $f(\theta) = c$. Hence, $\ker j_1 = \mathrm{span}\langle([\partial_\phi], 0), ([y\partial_x - x\partial_y], [\partial_\theta])\rangle \simeq \mathbb{C}^2$. Therefore,

$$H^1_\pi(SU(2)) \simeq \mathrm{span}\langle\partial_\theta, \partial_\phi\rangle = \mathbb{C}^2.$$

One could interpret this result by noting that the leaf space of the Poisson structure of $SU(2)$ consists of two circles N and T. Each point of N is a zero-dimensional leaf. Each point of T represents a single two-dimensional leaf. The open sets of the leaf space are $\{U, U\cup T : U \subset N \text{ open in the usual topology}\}$. The first cohomology of our Poisson structure is the "tangent space" to the leaf space. There are two "tangent directions", corresponding to the rotations of T and N.

5.5.3 The second cohomology.

From the Mayer-Vietoris exact sequence we have $H^2_\pi(M) \simeq \mathrm{Im}\delta_1 \oplus \mathrm{Im} i_2$. By exactness, $\mathrm{Im} i_2 \simeq \ker j_2$. Since for $a, b, c \in \mathbb{C}$

$$j_2([(ax^2 + bxy + cy^2)\partial_x \wedge \partial_y]_U, 0) =$$
$$= [(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta) r\partial_r \wedge \partial_\theta]_{U\cap V} = 0 \quad \Leftrightarrow \quad a = b = c = 0,$$

and $j_2([(ax\partial_x + by\partial_x + cx\partial_y + dy\partial_y)\wedge\partial_\phi]_U, 0) = 0$ for all a, b, c, d, it follows that

$$\ker j_2 \simeq \mathrm{span}\langle x\partial_x, y\partial_x, x\partial_y, y\partial_y\rangle \wedge \partial_\phi =$$
$$= \mathrm{span}\langle \pi_{SU(2)}, \partial_\theta\wedge\partial_\phi, (r\cos 2\theta\cdot\partial_r - \sin 2\theta\cdot\partial_\theta)\wedge\partial_\phi, (r\sin 2\theta\cdot\partial_r + \cos 2\theta\cdot\partial_\theta)\wedge\partial_\phi\rangle.$$

We now turn to $\mathrm{Im}\delta_1$. Since $\ker\delta_1 = \mathrm{Im}\, j_1$ and from (5.5.1)-(5.5.3) it follows that

$$\mathrm{Im}\, j_1 \simeq \mathfrak{X}^1(\theta) \oplus \mathrm{span}\langle 1, \cos 2\theta, \sin 2\theta\rangle \cdot r\partial_r.$$

Therefore,

$$H^1_\pi(U\cap V)/\ker\delta_1 = \{\, f(\theta)r\partial_r \,|\, f(\theta)\in C^\infty(\theta),\ f\perp \operatorname{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\}.$$

Recall that by definition $\delta_1([v])$ for $[v]\in H^1_\pi(U\cap V)$ is given by

$$\delta_1(v) = \begin{cases} [d_\pi X_U] & \text{on } U,\\ [d_\pi X_V] & \text{on } V,\end{cases}$$

where $X_U\in\mathfrak{X}^1(U)$, $X_V\in\mathfrak{X}^1(V)$ are any vector fields satisfying $(X_U - X_V)|_{U\cap V} = v$.

Let $v = f(\theta)r\partial_r$ and choose $X_U = \xi(r)f(\theta)r\partial_r$, $X_V = 0$, where $\xi(r)$ is a smooth function such that

$$\xi(r) = \begin{cases} 0 & \text{on } [0,1/6],\\ \text{strictly increasing} & \text{on } (1/6,1/3),\\ 1 & \text{on } [1/3,2/3].\end{cases}$$

(We need $\xi(r)=0$ near zero to guarantee that X is smooth on U; $\xi(r)=1$ on $[1/3,2/3]$ guarantees that $(X_U - X_V)_{|U\cap V} = v$). Since $\pi_U = r\partial_r\wedge\partial_\phi$, we obtain

$$\delta_1(v) = \begin{cases} d_\pi(\xi(r)\cdot f(\theta)r\partial_r) = \zeta(r)\cdot f(\theta)r\partial_r\wedge\partial_\phi & \text{on } U,\\ 0 & \text{on } V,\end{cases}$$

where $\zeta(r) = -\frac{d\xi(r)}{dr}$ is a "bump function" with support on $(1/6,1/3)$. Therefore,

$$\begin{aligned} H^2_\pi(SU(2)) = \operatorname{span}\langle \pi_{SU(2)},\ \partial_\theta\wedge\partial_\phi,\ (r\cos 2\theta\cdot\partial_r - \sin 2\theta\cdot\partial_\theta)\wedge\partial_\phi,\ (r\sin 2\theta\cdot\partial_r + \cos 2\theta\cdot\partial_\theta)\wedge\partial_\phi\rangle \\ \oplus\{\zeta(r)f(\theta)\cdot\pi_{SU(2)} : f(\theta)\perp\operatorname{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\}\end{aligned}$$

and is, in particular, infinite-dimensional.

5.5.4 The third cohomology.

From the Mayer-Vietoris exact sequence we have $H^3_\pi(M)\simeq \operatorname{Im} i_3\oplus\operatorname{Im}\delta_2\simeq H^3_\pi(U)\oplus\operatorname{Im}\delta_2$. To compute $\operatorname{Im}\delta_2$, we need to determine $\ker\delta_2\simeq\operatorname{Im} j_2$. First,

$$j_2\left(\left[\operatorname{span}\langle x,y\rangle\cdot\operatorname{span}\langle\partial_x,\partial_y\rangle\cdot\partial_\varphi\right],0\right) = 0$$

and for $a,b,c\in\mathbb{C}$

$$\begin{aligned} &j_2([ax^2+bxy+cy^2)\partial_x\wedge\partial_y]_U,0) = \\ &(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta)r\partial_r\wedge\partial_\theta.\end{aligned}$$

Therefore,

$$\frac{H^2_\pi(U\cap V)}{\mathrm{Im}\, j_2} = \{\, f(\theta)\cdot r\partial_r\wedge\partial_\theta : f(\theta)\in C^\infty(\theta),\ f(\theta)\perp \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\}$$

and

$$\mathrm{Im}\delta_2 = \{\zeta(r)\cdot f(\theta) r\partial_r\wedge\partial_\theta\wedge\partial_\phi : f(\theta)\perp \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle,$$

where $\zeta(r)$ is again a bump function concentrated on $(1/6,1/3)$. We obtain

$$H^3_\pi(SU(2)) \simeq (\mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle \oplus \{\zeta(r)f(\theta) : f((\theta)\perp \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\})\cdot r\partial_r\wedge\partial_\theta\wedge\partial_\phi.$$

The results of this chapter are summarized in the following

Theorem 5.5.1. *The Poisson cohomology of the standard r-matrix structure on $SU(2)$ is given by*

$$\begin{aligned}
H^0_\pi(SU(2)) &\simeq \mathit{span}\langle 1\rangle = \mathbb{C}^1\\
H^1_\pi(SU(2)) &\simeq \mathit{span}\langle \partial_\theta,\partial_\phi\rangle = \mathbb{C}^2\\
H^2_\pi(SU(2)) &\simeq \mathit{span}\langle \pi_{SU(2)},\partial_\theta\wedge\partial_\phi,(r\cos 2\theta\partial_r-\sin 2\theta\partial_\theta)\wedge\partial_\phi,(r\sin 2\theta\partial_r+\cos 2\theta\partial_\theta)\wedge\partial_\phi\rangle\oplus\\
&\quad \oplus\{\zeta(r)f(\theta)\cdot\pi_{SU(2)} : f(\theta)\perp \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\}\\
H^3_\pi(SU(2)) &\simeq \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle\cdot\pi_{SU(2)}\wedge\partial_\theta\oplus\\
&\quad \oplus\{\zeta(r)f(\theta)\cdot\pi_{SU(2)}\wedge\partial_\theta : f(\theta)\perp \mathrm{span}\langle 1,\cos 2\theta,\sin 2\theta\rangle
\end{aligned}$$

where $\zeta(r)$ is a bump function with support on an interval inside of $(0,1)$.

Proposition 5.5.2. *The Schouten bracket on multivector fields induces the following bracket on the Poisson cohomology:*

$$\begin{aligned}
&[[\partial_\phi],H^*_\pi(SU(2))]=0\\
&[[\partial_\theta],[r\cos 2\theta\partial_r\wedge\partial_\phi-\sin 2\theta\partial_\theta\wedge\partial_\phi]]=-2[(r\sin 2\theta\partial_r\wedge\partial_\phi+\cos 2\theta\partial_\theta\wedge\partial_\phi)]\\
&[[\partial_\theta],[r\sin 2\theta\partial_r\wedge\partial_\phi+\cos 2\theta\partial_\theta\wedge\partial_\phi]]=2[(r\cos 2\theta\partial_r\wedge\partial_\phi-\sin 2\theta\partial_\theta\wedge\partial_\phi)]\\
&[[\partial_\theta],[\zeta(r)f(\theta)\pi_{SU(2)}]]=\left[\zeta(r)\frac{df}{d\theta}\pi_{SU(2)}\right]\\
&[[\partial_\theta],[\cos 2\theta\cdot\pi_{SU(2)}\wedge\partial_\theta]]=-2[\sin 2\theta\cdot\pi_{SU(2)}\wedge\partial_\theta]\\
&[[\partial_\theta],[\sin 2\theta\cdot\pi_{SU(2)}\wedge\partial_\theta]]=2[\cos 2\theta\cdot\pi_{SU(2)}\wedge\partial_\theta]\\
&[[\partial_\theta],[\zeta(r)f(\theta)\cdot\pi_{SU(2)}\wedge\partial_\theta]]=\left[\zeta(r)\frac{df(\theta)}{d\theta}\cdot\pi_{SU(2)}\wedge\partial_\theta\right]\\
&[H^2_\pi(SU(2)),H^2_\pi(SU(2))]=0.
\end{aligned}$$

Proposition 5.5.3. *The wedge-product on the Poisson cohomology of $SU(2)$ is given by*

$$\begin{aligned}
&[\partial_\theta]\wedge[\partial_\phi] = [\partial_\theta\wedge\partial_\phi]\\
&[\partial_\theta]\wedge[\pi_{SU(2)}] = [\partial_\theta\wedge\pi_{SU(2)}]\\
&[\partial_\theta]\wedge[(r\cos 2\theta\partial_r - \sin 2\theta\partial_\theta)\wedge\partial_\phi] = [r\cos 2\theta\cdot\partial_r\wedge\partial_\phi\wedge\partial_\theta]\\
&[\partial_\theta]\wedge[(r\sin 2\theta\partial_r + \cos 2\theta)\wedge\partial_\phi] = [r\sin 2\theta\cdot\partial_r\wedge\partial_\phi\wedge\partial_\theta]\\
&[\partial_\theta]\wedge[\zeta(r)f(\theta)\cdot\pi_{SU(2)}] = [\zeta(r)f(\theta)\cdot r\partial_r\wedge\partial_\phi\wedge\partial_\theta].
\end{aligned}$$

5.6 Remarks.

5.6.1 Deformations associated to certain elements of the second cohomology.

Parameterize the two-dimensional symplectic leaves of $SU(2)$ by the argument of w; thus write D_θ for the leaf on which $\arg w = \theta$. Also let N be the set of zero-dimensional leaves (i.e., points with $w = 0$). It is not hard to check that $X_\theta = D_\theta \cup D_{\pi+\theta} \cup N$ is smooth and is isomorphic to a 2-sphere. The Poisson structure on this 2-sphere vanishes linearly along N. The two-dimensional leaves are precisely $D_{\pm\theta}$ and the zero-dimensional leaves are points in N. Such structures were studied in Chapter 3. Each such structure has an invariant called the regularized Liouville volume (see §3.2.5). The regularized volume of X_θ, as a function of θ, is an invariant of a Poisson structure having $D_\theta, D_{\pi+\theta}$, $0 \le \theta \le 2\pi$ as its two-dimensional leaves. For the standard Poisson structure on $SU(2)$, the regularized volume invariant of each X_θ is always zero because of the symmetry between D_θ and $D_{\pi+\theta}$. However, if we replace the Poisson structure $\pi_{SU(2)}$ by $\pi_{SU(2)} + \zeta(r)f(\theta)\pi_{SU(2)}$, where ζ a bump function as before and $f \perp \operatorname{span}\langle 1, \cos 2\theta, \sin 2\theta\rangle$, the leaves of the new Poisson structure stay the same, but the symplectic structure on each leaf is changed by "adding" more symplectic volume to D_θ and "subtracting" some symplectic volume from $D_{\pi+\theta}$; the net amount added to the regularized difference of volumes on D_θ and $D_{\pi+\theta}$ depends on $f(\theta) - f(\pi+\theta)$. Therefore, deformations of the kind considered above precisely change the values of the regularized symplectic volume of X_θ by an amount depending on $f(\theta) - f(\pi+\theta)$. This explains why deformations of the form written above for different $F(\theta) \doteq f(\theta) - f(\pi+\theta)$ are different in cohomology. It is rather mysterious why there are more deformations than just these. We mention that given an element in the second Poisson cohomology represented by $\zeta(r)f(\theta)\pi_{SU(2)}$, one can look for its antiderivative in the form

$$X = \xi(r)(f(\theta)r\partial_r) + g(\theta)\partial_\theta,$$

where $\xi(r)$ is such that $\xi'(r) = \zeta(r)$. Indeed, we have

$$d_\pi X = \zeta(r) f(\theta) r \partial_r \wedge \partial_\varphi.$$

Thus the issue is whether X is smooth on all of $SU(2)$. Because we can choose $\zeta(r)$ to be zero for large values of r, we may assume that X is supported on U. Moreover, since the only possible singularity of X occurs as $r \to 0$, it is enough to consider the region where $\zeta(r) = \text{const}$. Hence the question becomes whether, given a smooth periodic function $f(\theta)$, there exists a smooth periodic function $g(\theta)$, so that

$$f(\theta) r \partial_r + g(\theta) \partial_\theta$$

is a smooth vector field on $\mathbb{R}^2$ endowed with polar coordinates (r, θ). A somewhat cumbersome argument (which is in fact the verification of the exactness of the Mayer-Vietoris sequence in this case) shows that this happens if and only if $f(\theta)$ lies in the linear span of $1, \cos 2\theta$ and $\sin 2\theta$. We don't know a geometric interpretation of this fact (which is perhaps why we don't understand the relevant invariant of Poisson structures on $SU(2)$ distinguishing all deformations corresponding to the functions f orthogonal to this linear span).

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