

Heuristics: The study of methods of solution

(How to Solve It, George Polya, 2nd Ed., p. vii)

1. **Drawing Water from the River:** How can you bring up from the river exactly 6 quarts of water when you have only two containers, a 4 quart pail and a 9 quart pail to measure with? (p. 226)

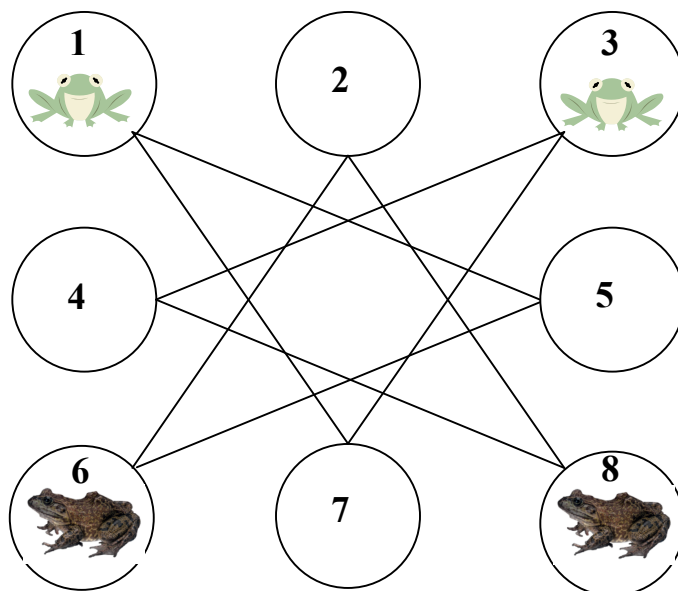
“Let us start from what is required and assume what is sought as already found. Let us inquire from what

antecedent the desired result could be derived.” Pappus (Greek mathematician, circa 300A.D.)

2. The Sailors and the Coconuts: Three sailors land on a deserted island where they find a pile of coconuts and a monkey. They agree to sleep overnight and divide the coconuts among themselves in the morning. During the night, one sailor wakes up, gives one coconut to the monkey, takes $\frac{1}{3}$ of the remaining coconuts and falls back asleep. Then a 2nd sailor wakes up and does the same. Later, the 3rd sailor wakes up and does the same. In the morning, there are fewer than 10 coconuts left, which the sailors divide evenly amongst themselves. How many coconuts were in the original pile?

3. **The Frog Problem:** In the illustration there are eight toadstools, with green frogs on 1 and 3 and brown frogs on 6 and 8. A “play” is to move one frog along one or more successive lines from toadstool to toadstool, never using an occupied toadstool. For example, if at some time there is a frog on 1 but not on 5 and 6, then it is just one play to move the frog from 1 to 5 to 6.

Can you use just seven plays to make the frogs exchange places, so that the green frogs end up on 6 and 8 and the brown ones on 1 and 3?



“To find a lucid geometric representation for your non-geometrical problem could be an important step toward the solution.” (Polya, p. 108)

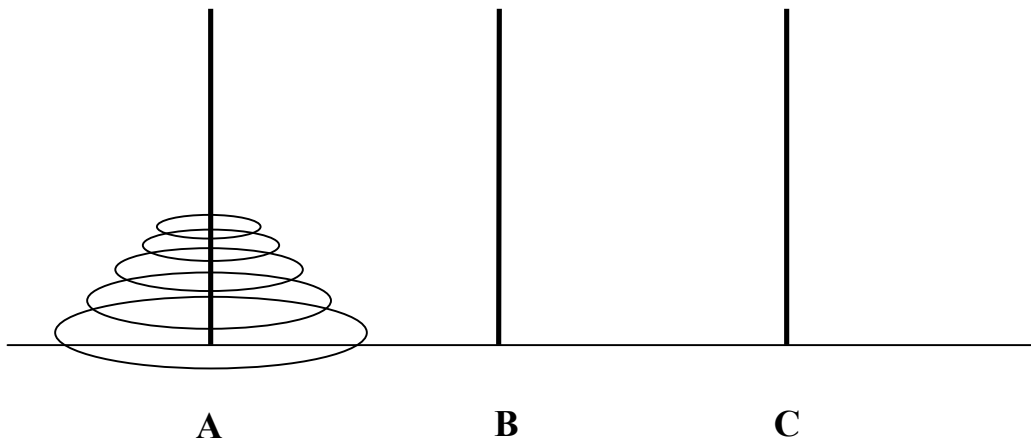
4. **Sum of the first n integers:** Find a formula for the sum of the first n integers. Develop a geometric and an algebraic solution. (Challenge: develop a second geometric solution.)

5. **Straightedge Construction:** Given Circle O , line l through its center, and point P not on l . Construct a perpendicular through P to l using only a straightedge.

“[Consider] a problem related to yours and solved before. Could you use it? Could you introduce some auxiliary element in order to make its use possible?”
(Polya, p. xvI)

6. Sum of the first n cubes: Find a formula for the sum of the first n cubes.

7. **Tower of Hanoi:** In the Tower of Hanoi toy, there are three posts. On one post (A) there is a stack of disks with different diameters, increasing in size downward like those in the illustration. The problem is to transfer all the disks to one of the other posts (B or C) in the fewest possible moves, *moving one disk at a time in such a way that no disk is ever placed on top of a smaller disk.*



What is the smallest number of moves required to transfer n disks from one post to another following the instructions?

Mathematical Induction (or passage to the next integer):

If:

- 1) You have an assertion dependent on integer n .
- 2) You are able to conclude the assertion is true for $n+1$ provided it is true for n .
- 3) You know assertion is true for $n = 1$.

Then, the assertion is true in general.

(Polya, p.120)