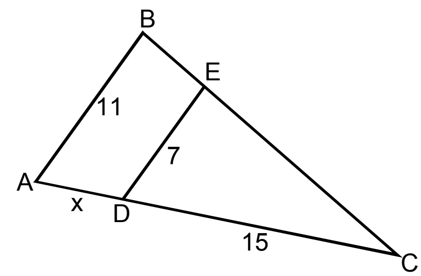
Solving Problems with Similar Triangles

<u>Example 1</u>: Given that lines DE and AB are parallel in the figure to the right, determine the value of x, the distance between points A and D.



Solution:

First, we can demonstrate that

CDE CAB

because

C = C (by identity)

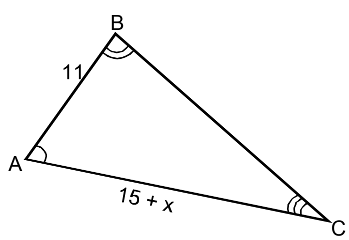
and

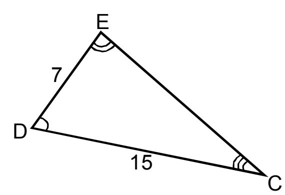
CDE = CAB

because line AC acts as a transversal across the parallel lines AB and DE, and since CDE and CAB are corresponding angles in this case, they are equal.

Since two pairs of corresponding angles are equal for the two triangles, we have demonstrated that they are similar triangles.

To avoid error in exploiting the similarity of these triangles, it is useful to redraw them as separate triangles:





Here we
have used a common technique for indicating corresponding angles between the two triangles:

the single arc in A and D indicate we intend to regard them as corresponding angles
the double arc in B and E indicate we intend to regard them as corresponding angles
the triple arc in C and C indicate we intend to regard them as corresponding angles

Note as well that we've labelled the sides with their lengths wherever possible. Now, according to property (ii) for similar triangles, we can write that

$$\frac{AC}{DC} = \frac{BA}{ED} = \frac{CB}{CE}$$

(Here AC means the length of the side between points A and C, etc.) We don't have any values or symbols for sides CB and CE, but substituting given lengths for the other four sides that appear here gives

$$\frac{15 + x}{15} = \frac{11}{7}$$

This gives an equation for x that is easy to solve. Solving,

$$15 + x = \frac{11}{7} \cdot 15$$

so

$$x = \frac{11}{7} \cdot 15 - 15 \approx 8.57$$

rounded to two decimal places.

Note that the equation coming from applying property (ii) involves complete triangle sides only. All four of the numbers in the equation we derived are to be lengths of complete sides. The property

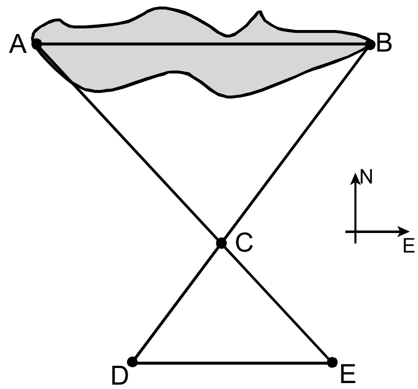
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

will give incorrect results if you use lengths of just parts of sides anywhere.

Also, once you've reviewed the notes in the basic trigonometry section, you will be able to tell that there is insufficient information here to be able to compute the value of x using the methods of trigonometry. So, in this example, the only effective method for determining x is to make use of the properties of similar triangles.

We'll give two more brief examples here to show how similar triangles can be used to solve somewhat more practical problems.

Example: Hank needs to determine the distance AB across a lake in an east-west direction as shown in the illustration to the right. He can't measure this distance directly over the water. So, instead, he sets up a situation as shown. He selects the point D from where a straight line to point B stays on land so he can measure distances. He drives a marker stick into the ground at another point C on the line between points D and B. He then moves eastward from point D to point E, so that the line of sight from point E to point A includes the marker stick at point C. Finally, with a long measuring tape, he determines that



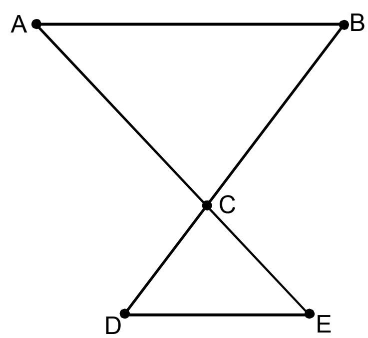
DE = 412 m
DC = 260 m
BC = 1264 m

and

CE = 308 m

Determine if this is enough information to calculate the distance AB, and if so, carry out the calculation.

solution:

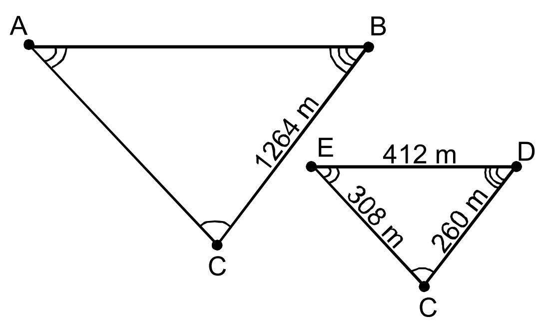


We've redrawn the diagram without the lake to make the geometry a bit easier to see. Lines AB and DE are both in an east-west direction, and so they must be parallel. Then lines AE and DB act as transversals across these parallel lines. This means that A = E and B = D (which form opposite interior angles in both cases). Thus

ABC   EDC

because

C = C
A = E
B = D



The figure to the right has the two triangles drawn in the same orientation to make it easier to keep track of corresponding angles and sides. The known lengths are included here. The property (ii) for similar triangles can now be written:

$$\frac{BC}{DC} = \frac{AC}{EC} = \frac{AB}{ED}$$

Substituting the known lengths into this triple-barrelled equation gives

$$\frac{1264\ m}{260\ m} = \frac{AC}{308\ m} = \frac{AB}{412\ m} \qquad (*)$$

Since all three of these ratios are equal, we can in particular equate the first and last to get

$$\frac{1264}{260} = \frac{AB}{412\ m}$$

(Notice that the units of m can be cancelled in the ratio on the left.) This gives an equation that can be solved to get AB:

$$AB = \frac{1264}{260} \times 412\ m = 2002.95\ m \approx 2003\ m$$

(Since all measured lengths here are rounded to the nearest meter, we should probably round at least that much in our answer.)

So, after all this analysis, we are able to determine that the desired distance from point A to point B across the lake is 2003 m.

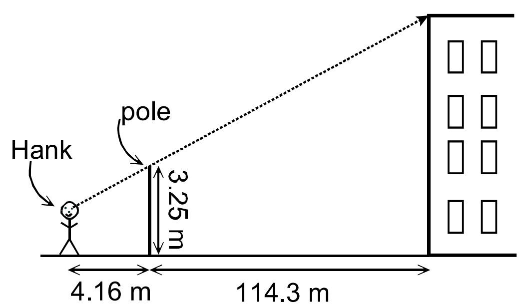
Hank didn't actually need to measure the length CE in order to be able to determine the length AB. However, having measured the length CE, he can also calculate the distance AC if he wishes (since line AC goes partially over water and so may not be measurable directly either, using the method Hank is using). From the relation (*) above, we can write

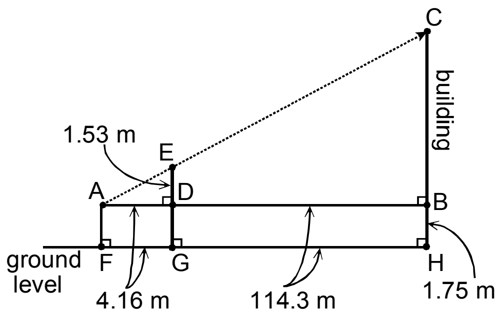
$$\frac{1264}{260} = \frac{AC}{308\ m}$$

so that

$$AC = \frac{1264}{260} \times 308\ m \approx 1497\ m$$

Example: Here's one more Hank story. Hank wishes to determine the height of a tall building. In the middle of a large level (horizontal) parking lot next to the building, there is a signpost whose top Hank measures to be 3.25 m above the ground. Hank then backs up from the post away from the building until the top of the post just lines up with the top of the building, and marks the spot where his feet are. Hank then





measures the distances as shown in the illustration. Finally, you should know that Hank's eyes are 1.72 m above the ground, when he stands straight up (as he did in this little experiment). Determine if this information is enough to calculate the height of the building, and if it is, do so for Hank.

Solution:

The geometry of the situation is shown in the second diagram to the right. There are a lot of lines and character labels here, but that will just make the following description clearer.

Point A is Hank's eye.
Point F is the spot on the ground where Hank stands.
Points E and G are the top and base of the signpost, respectively.
Points C and H are the top and base of the side of the building Hank is viewing.
The line FH is the horizontal ground level.
The line AB is a horizontal line through Hank's eye.

Our goal is to determine the height, CH, of the building.

We've also labelled known lengths in the diagram:

FG = 4.16 m by measurement
GH = 114.3 m by measurement
AF = 1.72 m by measurement

Then because AD and FH are both horizontal lines, the lengths DG and BH are the same as the length AF:

DG = BH = AF = 1.72 m

Since lines AF, EG, and CH are all vertical, we also have

AD = FG = 4.16 m

and

DB = GH = 114.3 m

Finally, since

ED + DG = 3.25 m

we can calculate that

ED = 3.25 m – DG
= 3.25 m – 1.72 m
= 1.53 m

Again, the goal is to compute the length CH. We can do this if we can determine the length CB, since

CH = CB + 1.72 m

This is beginning to look like a complicated problem. However, all we've been doing so far is sorting out the information provided and labelling the features of the situation very precisely. This preparation will now pay off in making it quite easy to complete the solution.

It is quite easy to demonstrate that

ADE   ABC

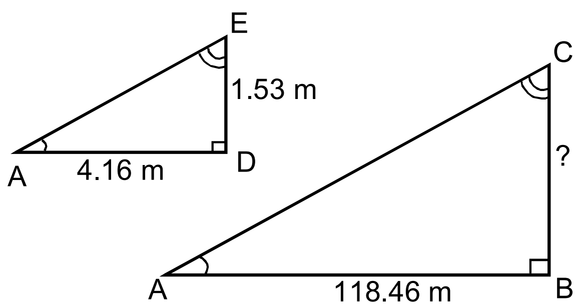
since

D =   B   (since both are right angles)
A =   A   (by identity)

As well

E =   C

because   E and   C are corresponding angles when we regard the line AC as a transversal across the parallel vertical lines EG and CH. Thus, we have the pair of similar triangles:



So, for these similar triangles, property (ii) of similar triangles says

$$\frac{AD}{AB} = \frac{ED}{CB} = \frac{AE}{AC}$$

or, in particular

$$\frac{4.16\ m}{118.46\ m} = \frac{1.53\ m}{CB}$$

Therefore

$$CB = \frac{(1.53\ m)(118.46\ \cancel{m})}{4.16\ \cancel{m}} @ 43.57\ m$$

But then

height of building = CH = CB + 1.72 m
= 43.57 m + 1.72 m
= 45.29 m

So, based on this data, we have computed that the building Hank is viewing is 45.29 m high.