Introduction to Induction (LAMC, 10/14/07)

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1 Definitions

The Method of Mathematical Induction (MMI) is usually stated as one of the axioms of the natural numbers (so-called Peano axioms), and thus, does not require proof.

Let $P(n)$ be a mathematical statement that depends on an integer n. (For example, $P(n)$ can be: "The number of primes which are smaller then n is less then $n/2$" or "$1 + 2 + \ldots + (n-1) + n = \frac{n(n-1)}{2}$").

One should think of $P(n)$ not as a single statement but as an infinite series of similar propositions: $P(1),\ P(2),\ P(3), \ldots$ for all integer values of n.

Principle of Mathematical Induction:
Suppose that

1. $P(1)$ *is true;*
2. *For any* $n \geq 1$ *"*$P(n)$ *is true" implies "*$P(n+1)$ *is true".*

Then $P(n)$ *is true for all* n.

To prove a statement using the Method of Math Induction, we need to complete two steps:

1. *Base case*: show that the statement $P(1)$ is true.
2. *Inductive step:* Assume that $P(n)$ is true. Show that this implies that $P(n+1)$ is also true.

We will solve problem 1 from page 3 as the first example.

Strong Induction

Sometimes when completing the inductive step (showing that $P(n+1)$ is true) we must use some (or all!) of the statements $P(1), P(2), \ldots P(n)$. This is called strong induction.

Suppose that

1. $P(1)$ *is true;*

2. *For any* $n \geq 1$, *"*$P(1)$, $P(2)$, $\ldots, P(n)$ *are true" implies "*$P(n+1)$ *is true".*

See problem 2 of section 5 for an example where you have to use the strong form of induction.

Other versions of induction

Sometimes you have to use a number other then 1 for the base of induction (e.g., in problem 2 of section 5).

Sometimes the base consists of several statements (e.g., in problem 3 of section 6).

2 A paradox: "All the horses are of the same color"

The following example was invented by George Polya to demonstrate that one has to be careful when applying the MMI.

Theorem. All horses are of the same color.

(Similarly, you can "proof" that, e.g., "all people are of the same height" or that "all test problems are equally difficult", etc.)

"Proof" (by induction).

1. *The base case:* consider a set consisting of one horse. In this set all horses are clearly of the same color (as there is just one horse).

2. *Inductive step:* Assume that the statement is true for n horses. (That is, in any set of n horses all horses are of the same color). Consider a set consisting of $(n+1)$ horse. Enumerate the horses in this set by number $1, \ldots, n+1$. Consider the following two sets of horses: $(1, 2, \ldots, n-1, n)$ and $(2, 3, \ldots, n, n+1)$. Each of the sets consists of n horses. By induction assumption, in each of the sets all the horses are of the same color. Since the two sets overlap, all the horses are of the same color.

The argument above obviously contains a mistake. Can you find it?

3 Proving identities using MMI

1. Let $S_n = 1 + \ldots + n$ (i.e., the sum of all integers starting from 1 and ending with n). Use induction to show that

$$S_n = \frac{n(n+1)}{2}.$$

(There is also a nice non-inductive proof. If you don't know it yet, try to think about it).

2. Let $S_n = \sum_{k=1}^{n} a_k$, where $a_k = a + (k-1)d$ for any k. (Note that S_n represents the sum of the first n elements of the arithmetic progression $\{a_k\}$). Can you guess the formula for S_n? Use induction to prove this formula.

3. Use induction to show that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. Use induction to show that

$$\left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

4 Divisibility results by Induction

1. Show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all n.
2. Show that $2^{3^n} + 1$ is divisible by 3^{n+1}.
3. Show that the number $111\dots111$ (written with 81 ones) is divisible by 81. (Can you find an inductive and non-inductive proof?)
4. Is it true that $n^2 + n + 41$ is prime for any natural number n?

5 Inequalities and Induction

1. Use induction to show that
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$
2. Use induction to show that $2^n > 1 + n \cdot \sqrt{2^{n-1}}$ for $n = 2, 3, \dots$. (Note that $n = 2$ should be used as the base of induction).
3. Show that
$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \leq \frac{1}{\sqrt{2n+1}}.$$

6 Elementary number theory and induction

1. Show that the equation $x^2 + y^2 = z^n$ has a solution in positive integers (x, y, z) for all $n = 1, 2, 3, \dots$. (*Hint:* consider the cases of n even and odd separately)
2. Prove that any natural number can be written as a sum of several distinct powers of two.(*Hint:* use **strong induction**).
3. Prove that one can pay any number of pesos bigger or equal to 7 using only the 3- and 5- peso notes.

7 Dividing and coloring the plane

1. The plane is divided into regions by several straight lines. Prove that one can color these regions using two colors so that any two adjacent regions have different colors. (Regions are adjacent if they have a common boundary).

2. Suppose that n lines are drawn on the plane in such a way that none of the lines are parallel and no three are meet at the same point. What is the number of the regions into which the lines divide the plane.

8 Challenge problems

1. Show that

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2 \cdot \cos\frac{\pi}{2^{n+1}},$$

   where n is the number of $\surd$ signs on the left.

2. Let F_k be Fibonacci numbers defined by: $F_0 = 0$; $F_1 = 1$, and if $k > 1$, $F_k = F_{k-1} + F_{k-2}$. Prove that

- $F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n$.
- $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.
- any positive integer can be written as a sum of distinct Fibonacci numbers.