Egyptian Multiplication

Beginners Circle 11/9/2014

Ancient Egyptians had an interesting method for multiplying two numbers. Suppose that you have to multiply two numbers (e.g., 23 and 18). The basic operation for them was multiplying a number by 2. (In other words, adding a number to itself). They reduced all other multiplication problems to it. Here is how they would start multiplying 23 by 18 (in modern notation):

23	18
1	**18**
2	**36**
4	**72**
8	144
16	**288**

Here is what they did to complete the multiplication

1. Below the first number (in this case, 23), they would write all of the powers of 2 that are smaller or equal to the first number.
2. In the second column, they would keep doubling the second number (in this case, 18).
3. After that, they would represent the first number as the sum of the powers of 2 (so that each of the powers of 2 is used at most once).
   For example, if the first number is 23, they would find

   $$23 = 16 + 4 + 2 + 1.$$

   After that, they would mark those rows where these powers of 2 are present in the left column. (In our example, the first, the second, the third, and the fifth rows are marked).

Finally, all there is to do at this point is to add the marked numbers in the second column:

$$\begin{array}{cccc} & & 1 & 8 \\ & & 3 & 6 \\ & & 7 & 2 \\ + & 2 & 8 & 8 \\ \hline & 4 & 1 & 4 \end{array}$$

Thus, the result of the multiplication is 414.

1. (a) Multiply 13 by 22 using Egyptian multiplication.

13	× 22
1	22
2	44
4	88
8	176

$13 = 8 + 4 + 1$

So, $13 \times 22 = 176 + 88 + 22$
$= 286$

(b) Find the binary representation of 13 using your computation in part (a).

1101

(c) Did you start with the highest or the lowest power of 2?

highest (left to right)

2. Explain how each number in the second column is obtained from the number in the first column, which is in the same row.

   E.g. How do you get

   - 72 from 4 and 18?

1	18
1	18
2	36
4	72

   - 144 from 8 and 18?

1	18
1	18
2	36
4	72
8	144

3. Using what you noticed in question 2 do the following:

   (a) Rewrite each term in the sum: $18 + 36 + 72 + 144$ as a product of 18 and a power of 2.

   - $18 =$ 18×2^0
   - $36 =$ 18×2^1
   - $72 =$ 18×2^2
   - $288 =$ 18×2^4

   (b) Rewrite the whole sum:

   - $18 + 36 + 72 + 288 =$ $18 \times 2^0 + 18 \times 2^1 + 18 \times 2^2 + 18 \times 2^4$

   (c) What do you notice? Can you simplify this expression by factoring out 18?

   Yes.

   $= 18(2^0 + 2^1 + 2^2 + 2^4)$

4. Multiply the following numbers using Egyptian Multiplication:

(a) 13×41

13	× 41
1	41
2	82
4	164
8	328

$13 = 8 + 4 + 1$ So, $13 \times 41 = 328 + 164 + 41$
$= 533$

(b) 41×13

41	× 13
1	13
2	26
4	52
8	104
16	208
32	416

$41 = 32 + 8 + 1$

$41 \times 13 = 416 + 104 + 13$
$= 533$

5. Given two numbers, which one (smaller or larger) will you use as the first number in Egyptian Multiplication? Why? Give an example to justify your answer.

Smaller, because the larger the first number, the more multiples we would need for the second number.

6. Explain in your own words how Egyptian Multiplication works.

Egyptian multiplication breaks down the first number into a sum of multiples of 2, and multiplies each of these to the second number.

It uses the distributive property of addition.

7. With a partner, have a race to see who can multiply numbers faster. One of you must use Egyptian Multiplication and the other must use regular, long multiplication. Race 6 times alternating the type of multiplication you do. Show your work below:

SOLUTIONS TO THE EGYPTIAN MULTIPLICATION METHOD

(a) 25×31

25	× 31
1	31
2	62
4	124
8	248
16	496

$25 = 16 + 8 + 1$

$25 \times 31 = 496 + 248 + 31$

$= 775$

(b) 38×45

38	× 45
1	45
2	90
4	180
8	360
16	720
32	1440

$38 = 32 + 4 + 2$

$38 \times 45 = 1440 + 180 + 90$

$= 1710$

(c) 12×63

12	× 63
1	63
2	126
4	252
8	504

$12 = 8 + 4$

$12 \times 63 = 504 + 252$

$= 756$

(d) 17×52

17×52

1	52
2	104
4	208
8	416
16	832

$17 = 16 + 1$

$17 \times 52 = 832 + 52$

$= 884$

(e) 112×85

85×112

1	112
2	224
4	448
8	896
16	1792
32	3584
64	7168

$85 = 64 + 16 + 4 + 1$

$112 \times 85 = 7168 + 1792 + 448 + 112$

$= 9520$

(f) 256×50

50×256

1	256
2	512
4	1024
8	2048
16	4096
32	8192

$50 = 32 + 16 + 2$

$256 \times 50 = 8192 + 4096 + 512$

$= 12800$

8. Do you like Egyptian Multiplication or long multiplication better? Why?

Long, because it is faster.

Russian Peasant Multiplication

Here is how Russian Peasant multiplication works in the case of 23×18:

23	18
23	**18**
11	**36**
5	**72**
2	144
1	**288**

The following is what we need to do to complete the multiplication:

1. In the first column, take the first number and continue dividing it by 2 until you get 1.
   - If there is a remainder, drop it. For example, $23 \div 2 = 11\ R1$, so we just write 11 in the next row.) The graphic below shows how to do this.

$$\begin{array}{ccccc}
23 & = & 11 \times 2 & + & 1 \\
11 & = & 5 \times 2 & + & 1 \\
5 & = & 2 \times 2 & + & 1 \\
2 & = & 1 \times 2 & + & 0 \\
1 & = & 0 \times 2 & + & 1
\end{array}$$

2. In the second column, we keep doubling the second number (in this case, 18).
3. After that, we choose the rows with odd numbers in the first column, and add up the corresponding numbers in the second column.

$$\begin{array}{cccc}
 & & 1 & 8 \\
 & & 3 & 6 \\
 & & 7 & 2 \\
+ & 2 & 8 & 8 \\
\hline
 & 4 & 1 & 4
\end{array}$$

Thus, the result of the multiplication is 414.

1. Multiply 13×22 using Russian Peasant Multiplication.

13	× 22
13	22
6	44
3	88
1	176

13 x 22 = 22 + 88 + 176
= 286

2. Using what you notice on the previous page, 1 do the following:

(a) Rewrite each term in the sum: $18 + 36 + 72 + 144$ as a product of 18 and another number:

- $18 =$ 18 x 1
- $36 =$ 18 X 2
- $72 =$ 18 x 4
- $288 =$ 18 X 16

(b) Rewrite the whole sum using the expressions above:

- $18 + 36 + 72 + 288 =$ 18 X 1 + 18 X 2 + 18x4 + 18 x 16

(c) What do you notice? Can you simplify this expression by factoring out 18?

= 18 (1 + 2 + + 4 + 16)
= 18 X 23

Note that in this algorithm we are starting with the lowest power of 2 present in the number 23.

3. Let's learn to write numbers as sums of powers of 2 (without repetition) *starting with the smallest power of* 2 by using Russian Multiplication. Recall how Russian Peasant Multiplication works for 23×18.

Is this row used?	23	18	Multiplication Factor
1	23	18	1
1	11	36	2
1	5	72	4
0	2	144	8
1	1	288	16

(a) Write the values in the column for 23 by using the following results of division by 2:

$$\begin{aligned} 23 \div 2 &= 11\ R\ 1 \\ 11 \div 2 &= 5\ R\ 1 \\ 5 \div 2 &= 2\ R\ 1 \\ 2 \div 2 &= 1\ R\ 0 \\ 1 \div 2 &= 0\ R\ 1 \end{aligned}$$

(b) Write the values in the column for 18 (each next value is the double of the previous value);

(c) In the column labeled "Is this row used?" write 1 if the row is used and 0 if it is not used in Russian Peasant Multiplication. (Remember that a row is used in the Russian Peasant Multiplication if the number in the first column is odd)

(d) What do the numbers in the column "Multiplication Factor" add up to (only using rows that are used)?

$1 + 2 + 4 + 16 = 23$

(e) Write down 23 as the sum of powers of 2 starting with the smallest power:

$23 = 2^0 + 2^1 + 2^2 + 2^4$

(f) Using your answers, write down the binary representation of 23.

10111

(start with the smallest power, go from right to left)

4. We now have an algorithm of writing a number as a sum of powers of 2 starting with the smallest power.

_	27	_
1	27	1
1	13	2
0	6	4
1	3	8
1	1	16

(a) At each step, divide the number in the middle column by 2 and write the result just below it (disregarding the remainder).

(b) If the number you got in the previous step is even, write 0 in the left column. If the number is odd, write 1 in the left column.

(c) In the right column, write the powers of 2, starting with $2^0 = 1$.

(d) Circle the powers of 2 that are in the rows where the left column has 1.

(e) The circled numbers in the right column add up to the number you started with.

(f) The numbers in the left column (starting with the top number) form the binary representation of the number you started with.

11011

5. To understand why this algorithm works, answer the following questions:

(a) What is the rightmost digit (0 or 1) in the binary representation of an even number?

0

(b) What is the rightmost digit (0 or 1) in the binary representation of an odd number?

1

6. Suppose you start with a binary number written with n digits. Using arithmetic operations, how do you get the number written by the first $(n-1)$ digits? (*Hint:* you might want to consider the cases of even and odd numbers separately). Explain the connection between the ideas you discovered in problems 5 and 6 and the algorithm of converting a number into binary notation using division by 2 that you saw in problem 4.

For even numbers:
If the original number is X, $(n-1)$ digits represent $\frac{X}{2}$.

For odd numbers, $\frac{X-1}{2}$

because you shift every digit by a power of 2.

7. Write the following numbers as sums of powers of 2 (binary notation) starting with the smallest power (Hint: Use the method of problem 4):

(a) $9 =$

$9 \div 2 = 4 R 1$ 1
$4 \div 2 = 2 R 0$ 0
$2 \div 2 = 1 R 0$ 0
$1 \div 2 = 0 R 1$ 1

Binary representation : 1001

(b) $14 =$

$14 \div 2 = 7 R 0$ 0
$7 \div 2 = 3 R 1$ 1
$3 \div 2 = 1 R 1$ 1
$1 \div 2 = 0 R 1$ 1

1110

(c) $23 =$

$23 \div 2 = 11 R 1$ 1
$11 \div 2 = 5 R 1$ 1
$5 \div 2 = 2 R 1$ 1
$2 \div 2 = 1 R 0$ 0
$1 \div 2 = 0 R 1$ 1

10111

(d) $44 =$

$44 \div 2 = 22 R 0$ 0
$22 \div 2 = 11 R 0$ 0
$11 \div 2 = 5 R 1$ 1
$5 \div 2 = 2 R 1$ 1
$2 \div 2 = 1 R 0$ 0
$1 \div 2 = 0 R 1$ 1

101100

8. Use the Russian Peasant multiplication in the following examples:

(a) $27 \times 23 =$

Is this row used?	27	23	Multiplication Factor
1	27	23	1
1	13	46	2
0	6	92	4
1	3	184	8
1	1	368	16

$23 + 46 + 184 + 368 = 621$

(b) $33 \times 22 =$

	33	22	
1	33	22	1
0	16	44	2
0	8	88	4
0	4	176	8
0	2	352	16
1	1	704	32

$= 704 + 22$

$= 726$

(c) $19 \times 45 =$

	19	45	
1	19	45	1
1	9	90	2
0	4	180	4
0	2	360	8
1	1	720	16

$= 45 + 90 + 720$

$= 855$

RACE!

11010

9. Represent each number below as a sum of powers of 2. One person uses the Egyptian method (which starts with the largest power of 2 first and multiplication by 2). The other person uses the Russian Peasant method (which starts with the smallest power of 2 first and division by 2).

   (a) 26

   EGYPTIAN

   1
   2
   4
   8
   16

   26 = 16 + 8 + 2

   so, 11010

   RUSSIAN PEASANT

26	0
13	1
6	0
3	1
1	1

   ↑ 11010

   (b) 45

   1
   2
   4
   8
   16
   32

   45 = 32 + 8 + 4 + 1

   so, 101101

45	1
22	0
11	1
5	1
2	0
1	1

   ⇒ 101101

   (c) 52

   1
   2
   4
   8
   16
   32

   52 = 32 + 16 + 4

   110100

52	0
26	0
13	1
6	0
3	1
1	1

   ⇒ 110100

   (d) 68

   1
   2
   4
   8
   16
   32
   64

   68 = 64 + 4

   1000100

68	0
34	0
17	1
8	0
4	0
2	0
1	1

   ⇒ 1000100

   (e) Which method do you like better, and why?

   Russian Peasant is faster.

10. Now we know three methods of multiplication: Egyptian Multiplication, Russian Peasant Multiplication, and the "normal" long multiplication.

   (a) Give an example where Egyptian Multiplication may be easier than the other two methods. Explain why.

   When one of the multiples is small & the other is easy to double.

   Eg. 13 X 25

   (b) Give an example where Russian Peasant Multiplication may be the easiest method, and explain why.

   If one of the multiples is a power of 2, division by 2 is easy.

   Eg. 16 X 7

   (c) Give an example where the regular long multiplication may be the easiest method, and explain why.

   Eg. 25679 X 10000

   Division & multiplication by 2 would be very slow if the numbers are very large.

11. Compare Russian Peasant Multiplication, Egyptian Multiplication and regular, long multiplication. What are the key differences? Which requires you to remember the most facts? Use the columns below to write the advantages for each type of multiplication.

Russian Peasant Multiplication	Regular, Long Multiplication	Egyptian Multiplication
Requires division.	Requires multiplication & addition	Requires multiplication & addition.
Requires facts.		
Faster.	Fastest.	Fast
	Easy to multiply large numbers.	

(answers can vary)