Theorem 1  Any two inscribed angles subtending the same arc on a circle are congruent.

Problem 1  Prove Theorem 1.
**Theorem 2** Let $AB$ and $CD$ be any two chords of a circle intersecting at the point $M$. Then $|AM| \times |MB| = |CM| \times |MD|$.

**Problem 2** Prove Theorem 2
A straight line segment \textit{bisector} is a straight line passing through the middle of the segment. A \textit{perpendicular bisector} is the bisector forming the right angle with the original line.

\textbf{Problem 3} \textit{Prove that a perpendicular bisector is the set of all the points in the plane equidistant from the ends of the segment. (Note that you need to prove two statements, not one!)}
**Theorem 3** Perpendicular bisectors of the sides of any triangle in the Euclidean plane have a common point. The point is the center of the circumference circumscribing the triangle.

The point is called the *circumcenter* of the triangle. The corresponding circle is called the *circumcircle*.

**Problem 4** Prove Theorem 3.
Theorem 4  The altitudes of any triangle in the Euclidean plane intersect at one point.

The point is called the orthocenter of the triangle.

We will need the following proposition as a tool to prove Theorem 4.

Proposition 1  If one takes a triangle in the Euclidean plane and draws a straight line parallel to the opposing side through every vertex of the triangle, then the sides of the original triangle are the midlines of the triangle formed by the straight lines.

Problem 5  Prove Proposition 1.
Problem 6  Use Theorem 3 and Proposition 1 to prove Theorem 4.
Introduction to Trigonometry

Consider a right triangle with the legs $a$, $b$, and the hypotenuse $c$. Let $x$ be the angle opposite to the side $a$.

Below we introduce four functions, $\sin x$ (reads sine of $x$), $\cos x$ (cosine of $x$), $\tan x$ (tangent of $x$), and $\cot x$ (cotangent of $x$) that are very important in science and engineering. Together with a few more of their siblings, they are called trigonometric functions. The part of math that studies their properties is called trigonometry.

$$\sin x = \frac{a}{c} \quad \cos x = \frac{b}{c} \quad \tan x = \frac{a}{b} \quad \cot x = \frac{b}{a}$$

**Problem 7** Consider a right triangle with the angle $x$ different from the triangle above. Prove that the values of $\sin x$, $\cos x$, $\tan x$, and $\cot x$ will be the same.
According to Problem 7, \( \sin x, \cos x, \tan x , \) and \( \cot x \) do not depend on the triangle. They are the functions of the angle \( x \) only.

Problem 8 Prove that \( \tan x = \frac{\sin x}{\cos x} \) and that \( \cot x = \frac{\cos x}{\sin x} \).

The following equation is known as the main trigonometric identity.

\[
\sin^2 x + \cos^2 x = 1 \quad (1)
\]

Problem 9 Prove (1).
Problem 10  Prove that \( \sin \left( \frac{\pi}{2} - x \right) = \cos x \) and that \( \cos \left( \frac{\pi}{2} - x \right) = \sin x \).

Problem 11  Express \( \tan \left( \frac{\pi}{2} - x \right) \) and \( \cot \left( \frac{\pi}{2} - x \right) \) as functions of \( x \), not \( \frac{\pi}{2} - x \).
Problem 12  Prove that the length of the leg of a right triangle opposite to the angle of $\pi/6$ equals half the length of the hypotenuse.
Problem 13  *Find the following values.*

- \( \sin \left( \frac{\pi}{6} \right) = \)
- \( \cos \left( \frac{\pi}{6} \right) = \)

- \( \tan \left( \frac{\pi}{6} \right) = \)
- \( \cot \left( \frac{\pi}{6} \right) = \)

- \( \sin \left( \frac{\pi}{3} \right) = \)
- \( \cos \left( \frac{\pi}{3} \right) = \)

- \( \tan \left( \frac{\pi}{3} \right) = \)
- \( \cot \left( \frac{\pi}{3} \right) = \)

- \( \sin \left( \frac{\pi}{4} \right) = \)
- \( \cos \left( \frac{\pi}{4} \right) = \)

- \( \tan \left( \frac{\pi}{4} \right) = \)
- \( \cot \left( \frac{\pi}{4} \right) = \)
Problem 14  Prove that the angular bisector is the set of all the points in the plane equidistant from the sides of the angle. (Note that you need to prove two statements, not one.)
Problem 15 Use the picture below as well as Problems 12 and 14 to find $\sin \left( \frac{\pi}{12} \right)$ and $\cos \left( \frac{\pi}{12} \right)$. 

![Diagram showing a right triangle with sides labeled as $a$, $2a$, and $\frac{\pi}{6}$]
If you are finished doing all the above, but there still remains some time ...

**Problem 16** While playing in the park, Ivan and Peter came to a large round clearing surrounded by a ring of aspen trees. The boys decided to count the trees. As they walked around the clearing counting, Ivan’s 20th tree turned out to be Peter’s 7th, while Ivan’s 7th was Peter’s 94th. How many trees were growing around the clearing?

**Problem 17** The length of the Earth’s equator is 40,075 km. A rope is tied tight around the Equator. Then they add a 1-meter-long piece to the rope and stretch it evenly above the Earth’s surface. Assuming that the Earth is a perfect ball, would a cat be able to squeeze itself between the Earth and the rope?
Problem 18  A train moves in one direction for 5.5 hours covering any 100-mile part of the journey in 1 hour. Is the train necessarily moving at a constant speed? Is the train’s average speed for the entire journey necessarily equal to 100 mph?