Let $O$ be a point in the plane, and $r > 0$ a fixed number. Call $C$ be the circle with center $O$ and radius $r$.

For each point $A \neq O$ in the plane, define the inverse of $A$, denoted by $A'$, the unique point on the half line $OA$ such that $OA \cdot OA' = r^2$.

Exercise 1) What is the image of the circle $C$ through inversion?

Exercise 2) What is the image of a line passing through $O$, through inversion?

Exercise 3) The inverse of a line not passing through $O$ is a circle passing through $O$.

Exercise 4) The inverse of a circle passing through $O$ is a line not passing through $O$. (DEDUCE THIS IMMEDIATELY FROM 3, without any other computations!)

Exercise 5) Suppose we have two circles, tangent at $O$. From Exercise 4, we know that the inverses of the two circles are two lines. Prove that these two lines are parallel.

Exercise 6) The inverse of a circle not passing through $O$ is a circle not passing through $O$.

Exercise 7) Where does the inverse of a point outside circle $C$ lie?

Exercise 8) Let $O$ be a point, and $\mathcal{D}$ a circle that does not contain $O$. Prove that there exists and $r > 0$, such that inversion of center $O$ and $r > 0$ keeps circle $\mathcal{D}$ invariant.

Exercise 9) (Appolonius’ circles - particular case) Suppose we have three circles, $C_1, C_2, C_3$, disjoint and none contains the others. Suppose $C_1$ and $C_2$ are tangent at a point $O$. Construct a circle $D$ that is tangent to all three of them.

Exercise 10) Suppose we have 4 circles, $C_1, C_2, C_3, C_4$, such that $C_1$ is tangent to $C_2$, $C_2$ to $C_3$, $C_3$ to $C_4$ and $C_4$ to $C_1$. Prove that four the tangency points form a cyclic quadrilateral.

Exercise 11) (AMC 12B, 2013, Prb 22): Let $m, n > 1$ be integers. Suppose that the product of the solutions for $x$ of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$. 

1