Comparing infinities

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Last time we learned how to count without going “one, two, three…” We saw that, by matching each student with one chair, and each chair with one student, we could say that there were exactly as many chairs as students.

This point of view will turn out to be very useful when we want to count infinite things. For instance, here are the natural numbers, with the even numbers in bold:

\[0, 1, 2, 3, 4, 5, 6, 7, 8, 9\ldots\]

Are there more naturals than evens, or just as many?

Here are the integers, with the natural numbers in bold:

\[\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\]

Are there more integers than naturals? What about fractions versus integers?

Since we can’t count all the naturals, or all the integers, or all the fractions, we can’t answer these questions by going “one, two, three…”; but we can answer them with bijections, that is, by trying to match every natural with an even, every integer with a fraction, etc.

**Some Simple Bijections.**

1. The positive integers are \(P = \{1, 2, 3, 4, \ldots\}\) and the negative integers are \(N = \{-1, -2, -3, -4, \ldots\}\). Can you find a bijection between \(P\) and \(N\)?

2. The even numbers are \(E = \{0, 2, 4, 6, \ldots\}\) and the odd numbers are \(O = \{1, 3, 5, \ldots\}\). Can you find a bijection between \(E\) and \(O\)?

3. The Hotel Infinity, which has rooms numbered \(1, 2, 3, 4, \ldots\) (forever) is full; every room has a guest in it. A traveler arrives at the lobby. If the manager is willing to move some guests around, can she find a place for the traveler? (Hint: find a bijection between the positive integers \(1, 2, 3, 4, \ldots\) and the natural numbers \(0, 1, 2, 3, 4, \ldots\); 0 plays the role of the traveler. Use this to describe what the manager needs to do with a simple English sentence.)

4. Can you find a bijection between the natural numbers \(N = \{0, 1, 2, 3, \ldots\}\) and the even numbers \(E = \{0, 2, 4, \ldots\}\)? If so, it has the surprising consequence that there are just as many even numbers as there are evens and odds combined!

5. At the Hotel Infinity, the rooms \(1, 2, 3, 4, \ldots\) have one guest each. A spacetime anomaly opens a doorway to a parallel universe, and the evil twin of every hotel guest suddenly appears in the lobby. Can the manager somehow accommodate these evil twins and the old guests, still with one person per room? (Hint: find a bijection between the nonzero integers \(\ldots, -3, -2, -1, 1, 2, 3, \ldots\) and the positive integers \(\{1, 2, 3, \ldots\}\). Here \(-n\) plays the role of the evil twin of the guest in room \(n\). Express the solution with a simple English sentence.)
Slightly Less Simple Bijections.

5*. Consider a grid of points which is infinite in all directions, like so:

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and a line of points which is infinite to the right, like so:

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Can you find a bijection between the points on the line and the points on the grid? If so, it has the surprising consequence that an infinite two-dimensional arrangement has as many points as an apparently smaller one-dimensional arrangement!

6. Explain how your bijection above gives a bijection between the natural numbers and all pairs of integers,
\[ \mathbb{Z} \times \mathbb{Z} = \{ \ldots, (-23, -17), \ldots, (0, -1), \ldots, (33, -5), \ldots, (7, 2), \ldots, (233, 55), \ldots \} \]

7*. An enemy submarine is somewhere on the number line. Right now it’s hidden underwater at some integer point which you don’t know. It’s also moving in some direction at constant speed; you don’t know the direction or speed either, only that it’s some integer number of units per minute. For instance, the sub could have started at 1 and be moving 3 units to the right per minute; or it could have started at −327 and be moving 22 units to the left per minute.

Every minute you can fire a missile at an integer of your choice. (It hits the integer without any delay.) If the the submarine is there, you hit it and it sinks. Can you devise a strategy for firing the missiles that is guaranteed to hit the sub at some point?

8. Imagine the positive fractions were laid out in a grid as follows:

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\[
\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \ldots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \ldots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \ldots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]
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Now imagine that, as in problem 5, you managed to find a bijection between the natural numbers and the positions on this grid. Does that give a bijection between the natural numbers and the positive nationals, i.e. the fractions?
The Calkin-Wilf Tree.

A binary tree is something that looks like this:

It has a root (the top node) and each node connects to two children below it. The name “binary” refers to the fact that every node has exactly two children. And yes, it is weird that the root of the tree is at the top. Deal with it.

Next we’ll look at a binary tree where the nodes are positive rational numbers, and there is a very simple rule for which rationals are the children of which. It’s called the Calkin-Wilf tree. The root is $\frac{1}{1}$, the simplest of all fractions; and each node $\frac{a}{b}$ has two children, $\frac{a}{a+b}$ on the left and $\frac{a+b}{b}$ on the right. Here are the first few levels:

It goes on forever. Draw the next level of the tree.

9. If a node on the C-W tree is labeled $\frac{22}{17}$, what are its left and right children?

11. Does $\frac{4}{1}$ appear in the C-W tree at all? Does $\frac{10}{1}$? What about $\frac{1}{12}$?

10. If a node on the C-W tree is labeled $\frac{5}{7}$, what is its parent? What is its grandparent?

12. Does $\frac{5}{7}$ actually appear anywhere in the C-W tree? Does $\frac{12}{19}$?

13*. On the first few levels of the C-W tree that we’ve drawn, all fractions are in lowest terms (that is, like $\frac{1}{2}$ rather than $\frac{2}{4}$). Does that happen forever? Why is that? (Hint: use induction. Assuming a node $\frac{a}{b}$ is in lowest terms, what can you say about its children?)

14**. Given a fraction $\frac{a}{b}$, how can you determine whether it shows up somewhere on the C-W tree? Do they all show up? (Hint: use induction on the quantity $a+b$ and notice that for a node’s parent that quantity is strictly smaller.)

15. Explain how the C-W tree gives a bijection between positive integers and positive rationals. (Hint: count the nodes of the tree level by level.)