Math Circle, HighSchool 2

November 6, 2014

Observation 1: Most of the problems can be done using a “double counting” method. This is done by counting things in two different ways, hence the results must be equal. For most of the problems, I gave the hint in class, by saying what you have to count in two different ways. If you skipped class and you need the hint, email me.

Observation 2: A different notation for combinations is \( \binom{n}{k} \). In these handouts I used \( C(n,k) \).

Observation 3: You might find this helpful (we proved them in class):

\[
(x + y)^n = \sum_{i=0}^{n} C(n,i)x^iy^{n-i}
\]

\[
C(n,k) = \frac{n!}{k!(n-k)!}
\]

Observation 4: When I say “Prove in two different ways”, I mean one algebraic proof, using the formulas above, and a more elegant double-counting proof.

1. A polynomial \( P(x) \) is such that \( P(0) = 1 \) and

\[
(P(x))^2 = 1 + x + x^{100} \cdot Q(x),
\]

where \( Q(x) \) is a polynomial. Show that the coefficient of \( x^{99} \) in the polynomial \( (P(x) + 1)^{100} \) is 0.

2. From now on, \( C(n,k) \) will represent the binomial coefficient. Prove that

\[
C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n
\]

\[
C(n, 0) - C(n, 1) + C(n, 2) - \ldots + (-1)^n C(n, n) = 0
\]

3. Show that the number of solutions to the equation

\[
a_1 + a_2 + \ldots + a_n = k
\]

with \( a_i \geq 0 \) and natural numbers is \( C(k + n - 1, k) \)

4. Let \( m, n \) be natural numbers, and \( p \) a prime number. Find the residue when you divide \( (m + n)^p - m^p - n^p \) to \( p \).
5. Prove, in two different ways (one, by using the formula for binomial coefficient, and the other, by using a counting argument) that \( C(n, k) = C(n - 1, k - 1) + C(n - 1, k) \)

6. Prove that, if \( n \geq q \) natural numbers, then

\[
\sum_{k=q}^{n} C(n, k)C(k, q) = 2^{n-q}C(n, q)
\]

7. Prove, using a double-counting proof, that \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \)

8. Prove, in two different ways, that \( C(2n, n) = C(n, 0)^2 + C(n, 1)^2 + \ldots + C(n, n)^2 \)

9. Prove that the number of paths along the edges of a grid starting from \((0, 0)\) and finishing in \((n, n)\) that do not cross the diagonal is equal to \(\frac{C(2n, n)}{n+1}\). These are called Catalan Numbers.

10.

11. Prove that the Catalan numbers, \( x_n = \frac{C(2n, n)}{n+1} \) satisfy the recurrence relation \( x_0 = 1 \) and

\[
x_{n+1} = \sum_{i=0}^{n} x_i x_{n-i}
\]