PERPENDICULAR BISECTORS, CIRCLES, AND MORE

MATH CIRCLE (INTERMEDIATE) 05/05/2013

In geometry, a perpendicular bisector of a given line segment is the line that

- is perpendicular to the line segment (that is, it intersects the segment at a right angle), and
- passes through the midpoint of the segment.

(1) Can you think of at least one reason why we might study perpendicular bisectors immediately after studying distance?

(2) Do you think the perpendicular bisector of a segment is unique? Explain. (This is important for proofs!)

(3) Consider a line segment $\overline{AB}$ and its perpendicular bisector l. Explain why (or prove) it must be true that if P lies on the perpendicular bisector of line segment $\bar{AB}$, then $PA = PB$.

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(4) Let P be a point such that $PA = PB$. Explain why (or prove) it must be true that P lies on the perpendicular bisector of AB.

(5) We say that the **geometric locus** (or simply locus) of points is the set of all points that satisfy a certain condition. Draw and explain the following loci:

(a) the locus of points that is equidistant from the two endpoints of a line segment $\overline{AB}$

(b) the locus of points that is equidistant from a point A

(6) Define the following terms:

(a) circle

(b) unit circle

(7) How many possible circles can you draw through any...
   (a) 1 given point?

   (b) 2 given points?

   (c) 3 given points? (Is there a special restriction you must place on these points?)

   (d) 4 or more given points? (Is there a special restriction you must place on these points?)

(8) Fun with circles, triangles, and perpendicular bisectors!

(a) Draw a triangle ABC with (arbitrary) side lengths approximately 2, 2, and 3 inches.

(b) Inscribe $\triangle ABC$ in a circle. (Use your diagram from (a).) Do you think you would be able to do this for *any* three points A, B, and C, provided that $\triangle ABC$ is nondegenerate?

(c) Draw into your diagram from (a) the perpendicular bisectors of the three sides of $\triangle ABC$. Do they intersect? Is the point special with respect to the circle?

(9) Hopefully the previous problem gave you something to think about. Now it's time to prove it!

(a) We will be able to inscribe triangle ABC in a circle if and only if there exists a 'suitable' point O to serve as the circle's center. Explain why we know that such a point O would be equidistant from the vertices A, B, and C.
(Hint: What can you say about the segments $\overline{AO}$, $\overline{BO}$, and $\overline{CO}$ in relation to the circle? Make a sketch!)

(b) Use your results from part (a) and your knowledge of perpendicular bisectors to explain how we know that point O lies simultaneously on the perpendicular bisectors of all three sides of the triangle. (Because two intersecting lines can intersect at only one point, we have also shown that point O must be unique!)

(c) Because we have shown that point O simultaneously lies on the perpendicular bisectors of all three sides, we have proven two important things. What are they?

(i) (This one should be about the perpendicular bisectors of the three sides of a triangle)

(ii) (This one should be about how many circles can be drawn through a set of three points)

(10) Now suppose $\triangle ABC$ is a degenerate triangle. Draw an example $\triangle ABC$ and draw the perpendicular bisectors of all three sides. What part of Problem 13 is no longer satisfied? Is it possible to draw a circle through the points A, B, and C?

(11) A given segment is moving, remaining parallel to itself, in such a way that one of its endpoints lies on a given circle. Find the geometric locus described by the other endpoint.

(12) A given segment is moving in such a way that its endpoints slide along the sides of a right triangle. Find the locus described by the midpoint of this segment.

(13) Now, let's try something called a **construction**, in which we will use a compass, straight edge, and pencil (and nothing else) to draw a specified object. Our goal will be to construct a line segment and its perpendicular bisector.

(a) We're going to first find enough points on the bisector so that we can draw the only line through those points. How many points will we need? Why?

(b) Can you think of a way to construct those points (using only the materials you have)?
(Note: if you're using a ruler as your straight edge, you are not allowed to use it to measure distance!)

(c) Now you're ready to go! Construct a line segment and its perpendicular bisector. Once you think you've got it, call over the assistant at your table and perform the construction a second time while they watch!

(14) (Challenge problem) Prove that the closest and farthest points of a given circle from a given point lie on the secant passing through this point and the center.
(Hint: Apply the triangle inequality!)