GAMES ON GRAPHS

BEGINNER CIRCLE 12/2/2012

1. Finding Lost Students

As you well know, at the end of every meeting the instructors have to search the corridors for lost students. The students wander the corridors at a rate of one room per minute. At the same time, the instructors also walk through the hallways at a rate of one hallway per minute. As the instructors are lazy, they wish to have the least number of instructors patrol each hallway.
We can represent this problem using graphs as follows. Consider a graph where the edges represent the hallways. With a partner at your table, play a game where one person plays as the student, and the other player plays as the instructors.
The instructor places his teachers at the big person on the graph, and the student places his piece on the smaller person. On each players' turn, they must move their pieces one edge over. The instructors win if they can catch the student, and the student wins if they are able to avoid being caught.
Obviously, the teachers can win by starting with an instructor in every room. What is the fewest number of instructors that they need to win?

Problem 1. How many teachers are needed to guard each of the following floors?

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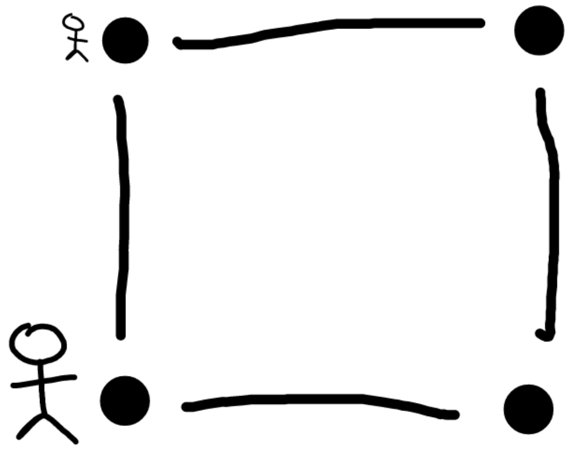
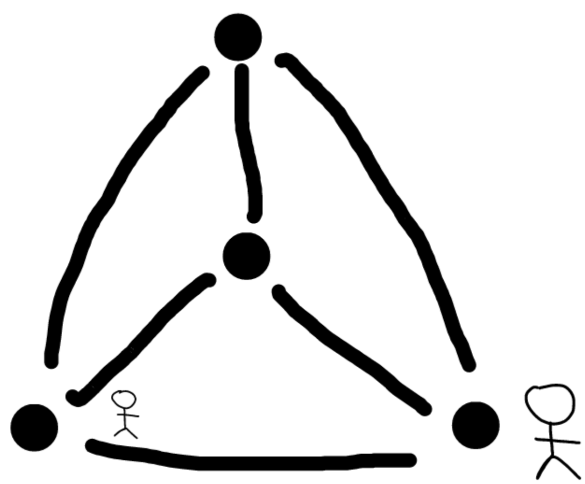
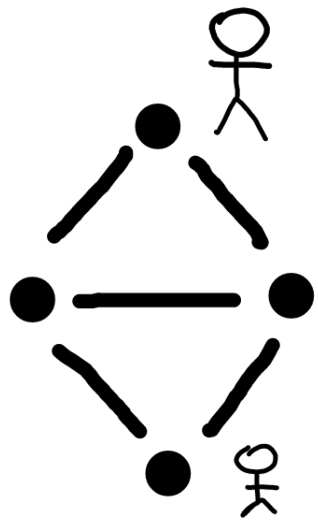
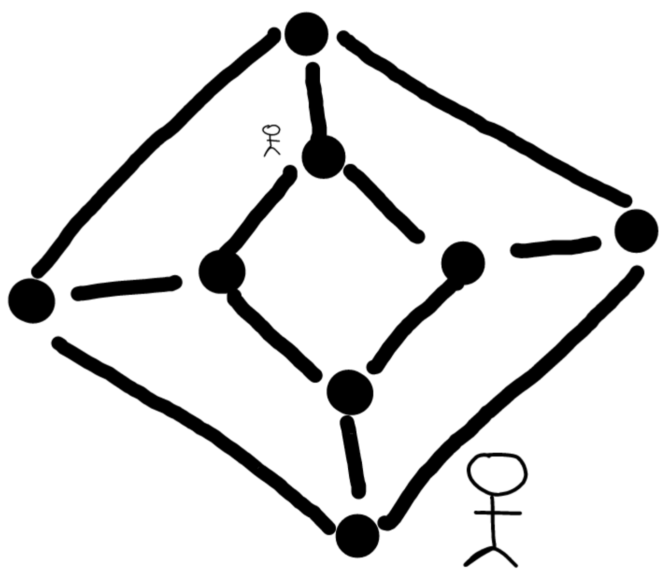
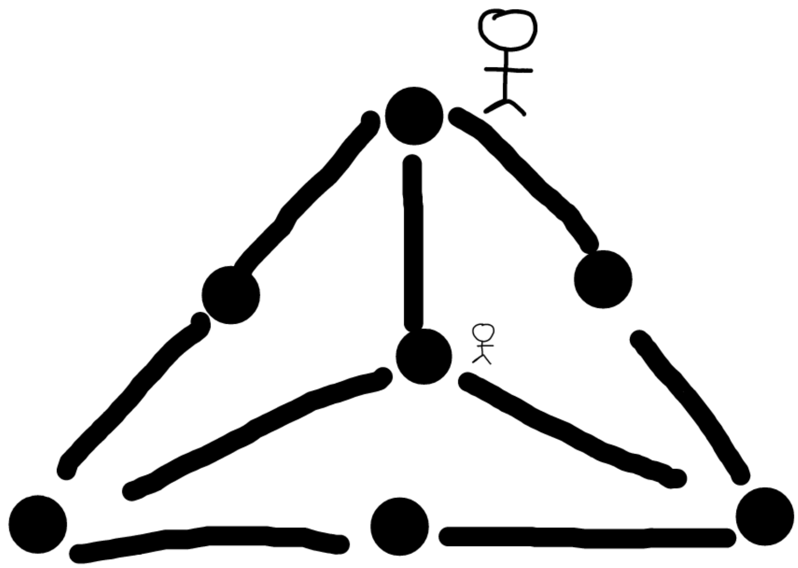










Problem 2. Can you come up with a floorplan that requires only 1 teacher to find the students if the teachers start in a certain place, but 2 teachers otherwise?

Problem 3. Can you come up with a floorplan (with starting positions) that requires 4 teachers to guard?

Problem 4. Can you design a floor plan that 1 teacher can cover if they move first, but requires 2 teachers if the student moves first?

Problem 5. Can you design a floor plan that has n vertices, but only needs 1 teacher to guard? Explain the design in full sentences.

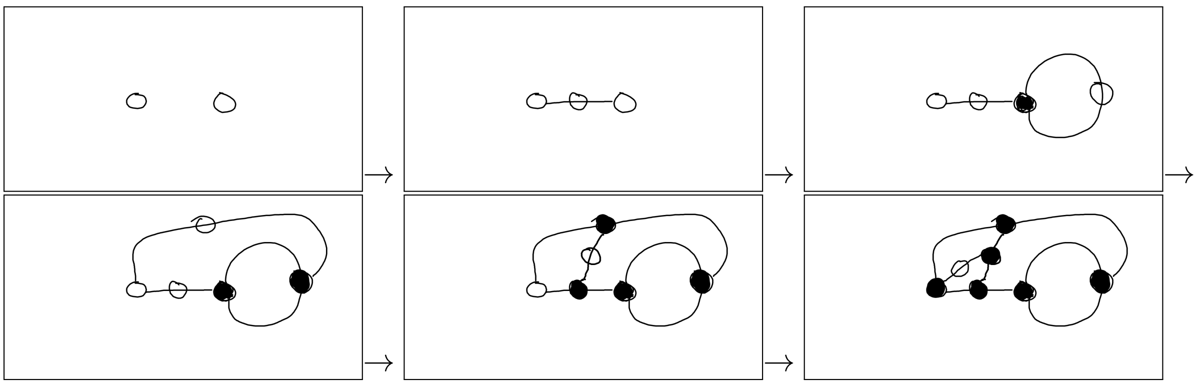
Problem 6. Can you design a floor plan (with starting places) that has 6 vertices and requires 3 teachers to guard? What about one with 8 vertices and 4 teachers to guard? $2n$ vertices and n teachers?

Problem 7. Can you design a floor plan that requires 3 teachers to guard, no matter where the student or teachers start?

2. Sprouts

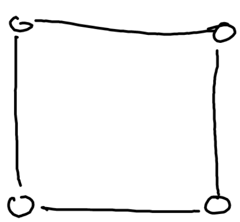
The rules of sprouts are as follows: We start with some spots drawn on a page.

- On your turn, you must connect two spots (or a spot to itself) with an edge. The edge must be drawn as not to cross any other edge already drawn. When you draw another edge, add a spot to the middle of the edge drawn.
- When a spot has 3 edges connected to it, it dies and can no longer accept new edges.
- When you cannot place an edge on your turn, you lose.



Problem 8. Can you show that this game always finishes?

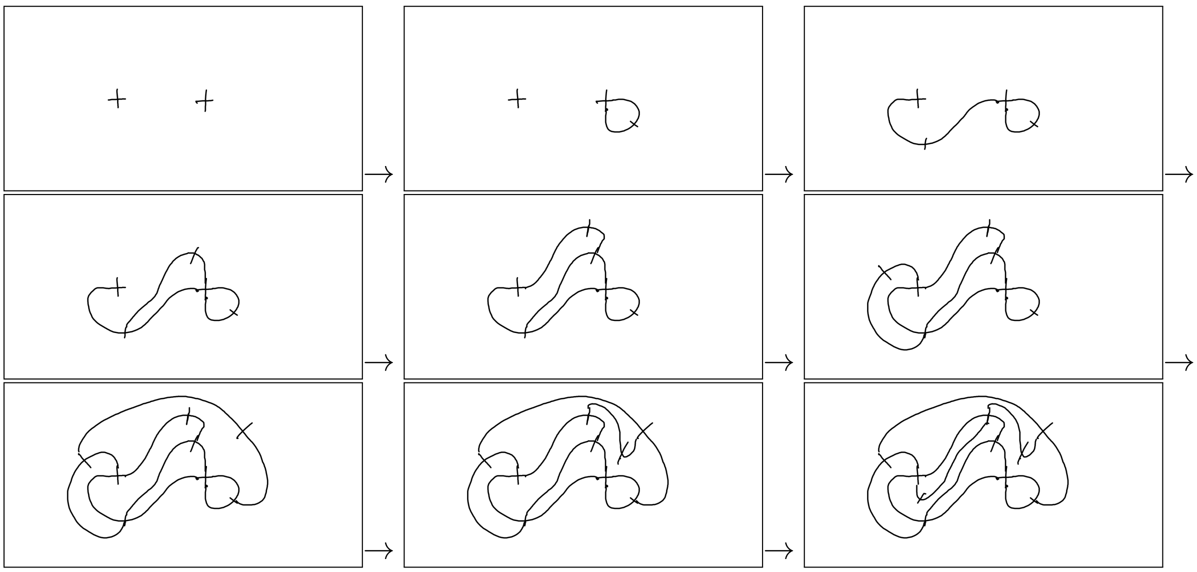
Problem 9. Can you show who wins when we start in this position?



3. Brussel Sprouts

The rules of Brussel Sprouts are as follows: We start with some crosses drawn on a sheet of paper. The tips of the crosses are called spikes.

- On your turn, you may connect two spikes with an edge. This edge may not cross any edge already drawn. When you have drawn an edge, add a short stroke across that edge to make two more spikes.
- Each spike may take at most 1 edge
- When you cannot place an edge on your turn, you lose.



Problem 10. Explain in full sentences why this game always finishes?

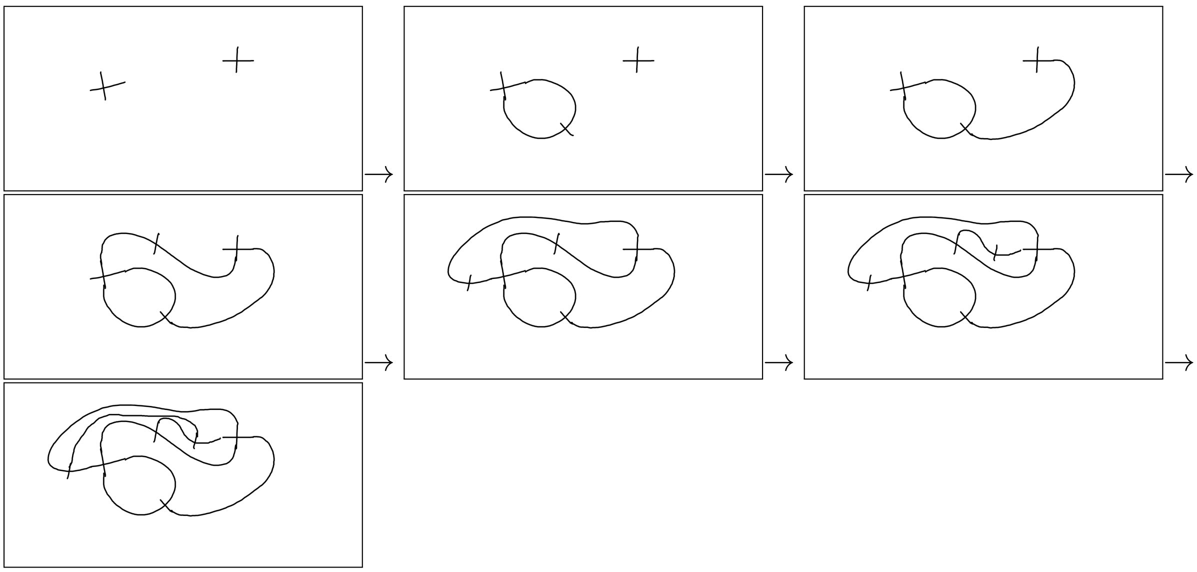
Problem 11. Can you show who has a winning strategy when you start with 2 crosses?

Problem 12. Can you show that this game always ends in $5n - 2$ turns, where n is the number of starting crosses?

4. Soy Sprouts

The rules of Soy sprouts are as follows: We start with some stars drawn on a sheet of paper. The tips of the crosses are called spikes.

- On your turn, you may connect two spikes with an edge. This edge may not cross any edge already drawn. When you have drawn an edge, you may, **if you wish**, add a short stroke across that edge to make two more spikes.
- Each spike may take at most 1 edge
- When you cannot place an edge on your turn, you lose.



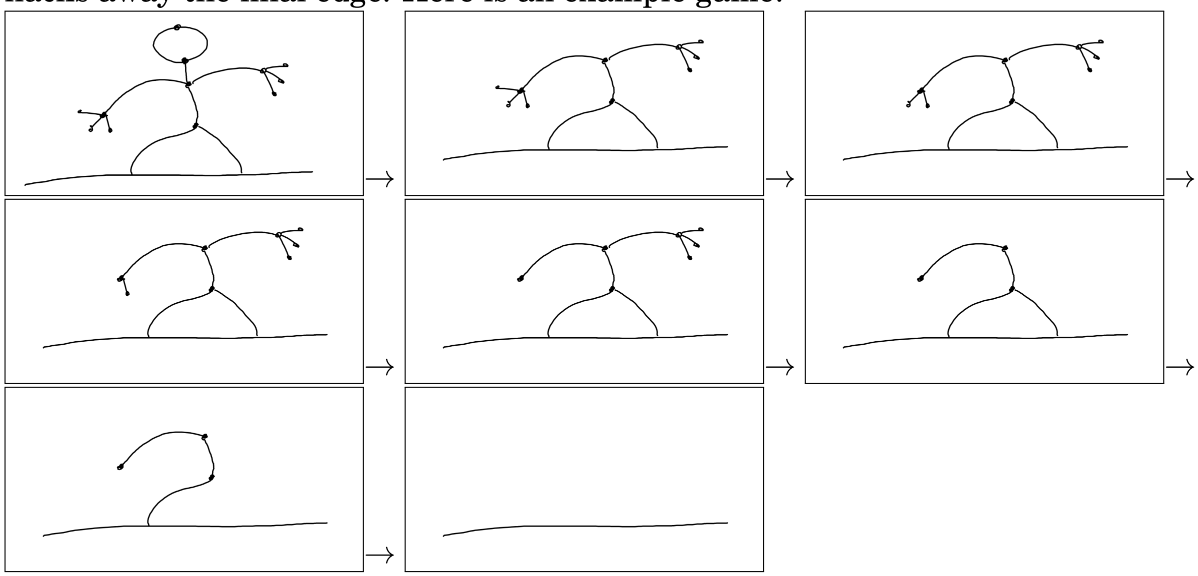
Problem 13. Can you show that a game always finishes?

Problem 14. Can you show who has a winning strategy when you start with 2 stars with 4 spikes each?

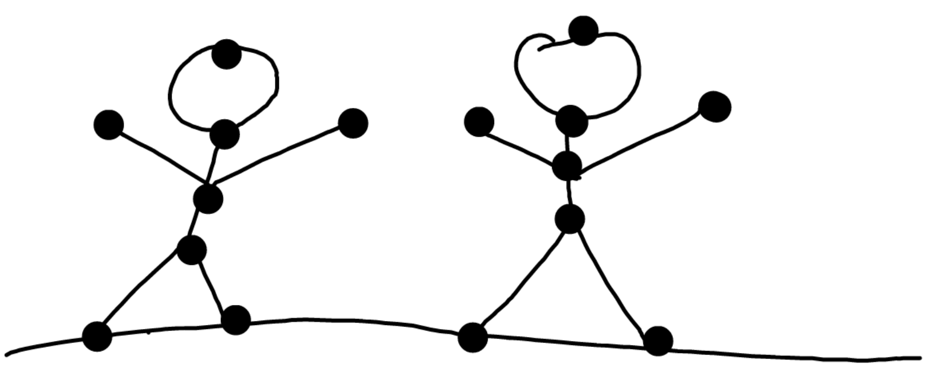
Problem 15. Can you show who has a winning strategy when we start with a single star with 4 spikes? what about 5 spikes? What about n spikes?

5. Hackenbush

In the game of Hackenbush, you start with a graph connected to a line called the ground. On your turn, you may remove one edge from the graph. Every edge that is not connected to the ground falls off the graph. The game is finished when a player hacks away the final edge. Here is an example game:



Problem 16. Why does second player always win this game of Hackenbush?



Problem 17. The game of Nim is a game where several piles of coins are on a table. On each player's turn, they are able to remove as many coins as they want from a single pile. The winner is the player who removes the last coin from the table.
Can you represent this game as a game of Hackenbush?