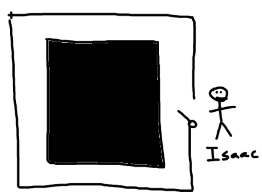
HANDSHAKING AND CHASING KIDS

BEGINNER CIRCLE 11/4/2012

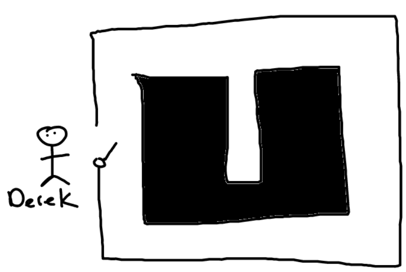
1. Kids running around Math Science

At the end of every day, Isaac, Derek, Jonathan, Jeff and Morgan have to check the hallways of Math Science to look for runaway students. As they are all lazy, they only want to walk down each corridor once as they look for students. So they take paths which are efficient: ones that walk down each corridor just once, and start and stop at the same spot. On which floors do the instructors get to be lazy?

Problem 1. Isaac always checks the 1st floor. Is it possible for him to check all the corridors without walking any corridor twice?

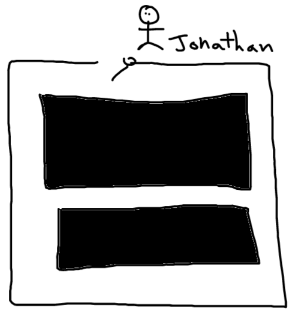


Problem 2. Derek always checks the second floor. Can you find a path that goes through every corridor exactly once? Why or why not?

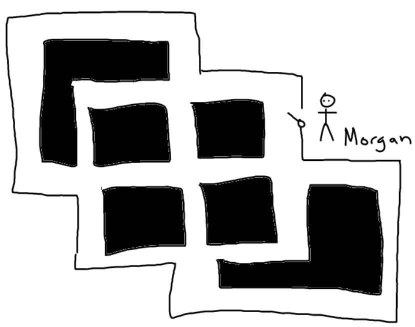


Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

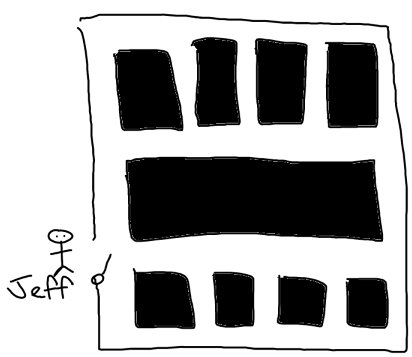
Problem 3. When Math Circle finishes, Jonathan runs up to the fifth floor (which, confusingly is the ground floor). Can Jonathan check all of the corridors without checking any corridor twice?



Problem 4. As Morgan is the fastest of all of the Math Circle instructors, he checks the largest floor (which happens to be floor seven). Can he check every corridor without going to any corridor twice?



Problem 5. Jeff decides to check the confusing eighth floor of Math Science. Can he do it efficiently?



2. Graphs

A **graph** is a collection of objects called **vertices** and a collection of **edges** that go in between them, which have the following properties:

- There is at most one edge connecting any two vertices
- Every edge connects two different vertices

Here are some examples of graphs and non-graphs

FIGURE 2.1. Things that are Graphs

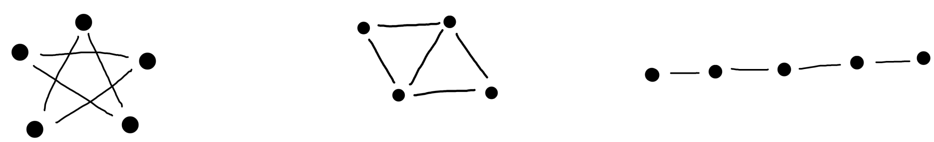
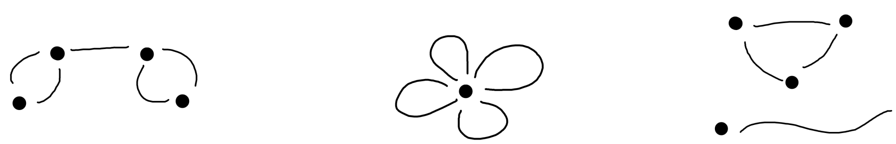
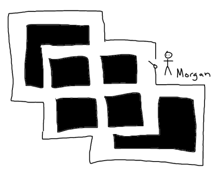


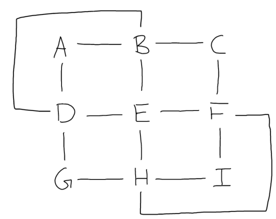
FIGURE 2.2. Things that are not Graphs



When we have a graph, one of the properties we are most interested in is the number of edges that connect to each vertex. If v is a vertex of a graph, then the **degree** of v (written $\deg v$) is the number of edges that connect to that vertex. If no edges connect to the vertex, we say it has degree 0.

Let us return to the problem with checking for students. These problems can be represented with graphs!
We can represent the places where the corridors intersect as vertices and represent the corridors themselves as edges. For example, the problem with Morgan's corridor looks like this:

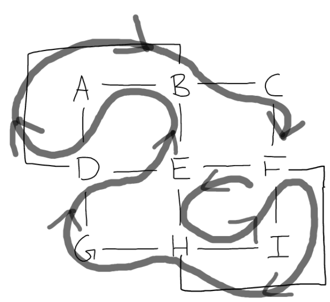




If we want to talk about a specific corridor in this graph, we can just say the two vertices that sit on the end of that corridor. For instance, Morgan is standing next to the corridor CF. It is important to notice that we could just as well say that Morgan is sitting next to the corridor FC. If we want to talk about a path that Morgan takes, we can just specify the order of the vertices that he visits. For example, the sequence

$$FEHIFHGDRBADBCF$$

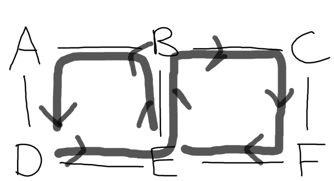
means the path



So a path is a sequence of vertices that satisfies these two properties

(1) Every adjacent pair of vertices in the sequence are connected by an edge
(2) No edge is used twice

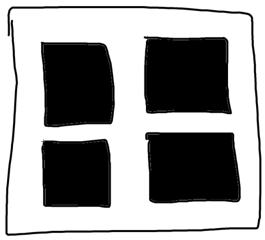
The above sequence is an example of a path. However, on the graph below, the following is not an example of a path:



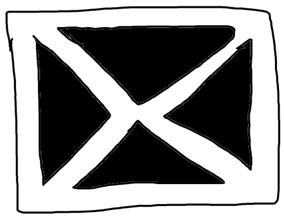
Because the edge BE is repeated in the sequence $DEBCFECA$.

Problem 6. Can you convert the following hallways into graphs? Label the vertices with letters.

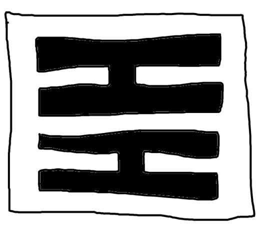
(a)



(b)



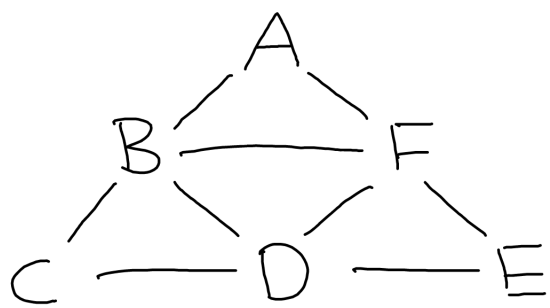
(c)



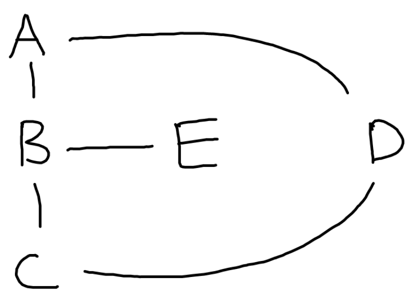
Problem 7. Go back to the original hallway problems. Convert the problem into graphs, and then write down the paths using a sequence of vertices.

Problem 8. Can you find a path that visits every edge only once on the graphs below? Write out the vertices that the path visits in order. It is ok if your path visits the same vertex multiple times.

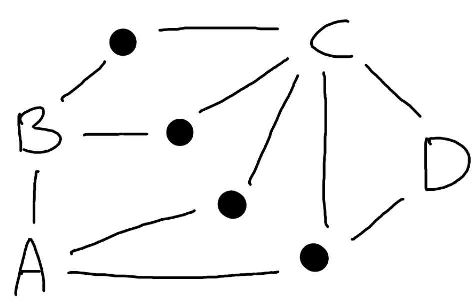
(a)



(b)



Problem 9. Find the degrees of the following vertices in this graph:



(a) What is $\deg A$?

(b) What is $\deg B$?

(c) What is $\deg C$?

(d) What is $\deg D$?

What is the sum of the degrees of this graph?

Problem 10 (Handshake lemma)**.** Isaac is having a party, and invites over his friends: Jeff, Morgan, Derek, and Jonathan. At the party, people shake hands many times. Isaac has an obsession with hand shaking, and he wants to know exactly how many handshakes have happened. He learns that Jeff shook hands with everybody besides Isaac, and Morgan and Jonathan shook hands as well. Isaac shook no hands, as he was so busy counting handshakes.

(a) Can you turn this problem into a graph? (Hint: Use the people as the vertices)

(b) How many handshakes occurred that evening?

(c) Isaac goes around and asks each person how many hands they shook, and then sums up those numbers. How many handshakes does Isaac count this way?

(d) Using the above as an inspiration, explain using complete sentences why the sum of the degrees of vertices in a graph is always twice the number of edges.

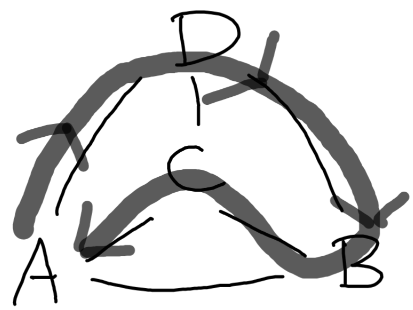
(e) Conclude that the sum of the degrees of vertices is always even.

3. Cycles

A **cycle** is a path that starts and ends at the same point. For example, the path

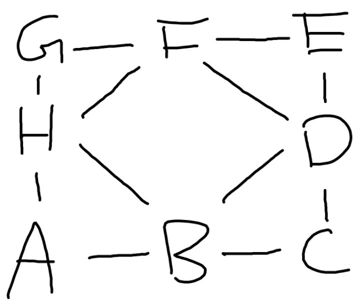
$$ADBCA$$

is a cycle in the graph

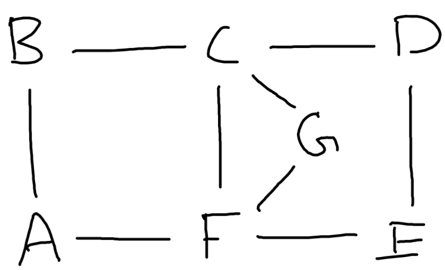


Problem 11. A cycle is called **Efficient** if it visits every edge. Can you find an Efficient cycle for the following graphs? Write down the cycle.

(a)



(b)



Problem 12. Last week we covered a concept, called Proof by Contrapositive. In this problem, we will prove that if a graph has an Efficient Cycle, then every one of its degree has even degree.

(a) Recall, the contrapositive the statement

  If A then B

 is the statement

  If (not A), then (not B)

 What is the contrapositive of the statement

  If a graph has an Efficient Cycle, then the degree of every vertex is even

 Use full sentences for your answer.

(b) If a vertex has odd degree, why can't every one of its edges belong to a cycle? (Hint: a cycle "enters" and "leaves" a vertex along its edges an equal number of times). Explain in a full sentence.

(c) Why does this show that if there is a vertex with odd degree, there are no Efficient cycles? Write your solution down in full sentences.

(d) Conclude that if a graph has an Efficient cycle, then all of its vertices have even degree.

Problem 13. Last week, we did a different kind of proof, called a **proof by contradiction**. In this proof, if we wanted to show a statment A, we first looked at its opposite. Then we showed that its opposite statement proved something absurd, like $0 = 1$

Morgan has an obsession with sorting things. He is having a party next week, and decides that he will rank the 50 guests by popularity. He will say that one person is more popular they have more friends at the party. Why is it that there are two people with the same number of friends?

(a) For a proof by contradiction, we first assume the opposite statement is true. What is the opposite statement of "There are two people at the party with the same number of friends at the party"? Use full sentences for your answer.

(b) Can you turn this problem into one about graphs? What are the vertices? What about the edges?

(c) Why does this tell us that there is a person who has 49 friends, and also show that someone at the party has no friends. Explain in full sentences,

(d) Why is the person with 49 friends is friends with everybody in the room? Why is this absurd?

(e) Conclude that there are two people that have the same number of friends.

Problem 14 (From *Graphs* by Gurowitz and Khovring). Can you write the digits 1-9 in a row so that the sum of adjacent values are divisible by 5, 7 or 13? For example, $4-9-1$ works, but $1-6-2$ does not. (Hint: Turn the problem into a graph! What should the vertices be? What about the edges?)

Problem 15 (Even implies an Efficient cycle). A graph is connected if there is a path connecting any 2 points. In problem 12 we showed that if a graph had an Efficient cycle, then all of the vertices had even degree. We will now show the reverse: that if a graph is connected and has all even degree vertices, then it has an efficient cycle. We will do this as a proof by contradiction.

(a) Remember for a proof by contradiction, we assume the opposite of what we want to prove, and then arrive at an absurd conclusion. What is the opposite of "If a graph has all even degree vertices, then it has an Efficient cycle"? Use a full sentence

(b) Let P be the longest path that uses no edge twice in our graph start at a vertex A and end at a vertex B. How is it that we know that $A = B$. (Hint: use the fact that every vertex has even degree)

(c) Suppose that this path does not contain every single edge. Then look at the graph missing all of the edges in P. Why does this new graph also have all even degree vertices? Use complete sentences to explain.

(d) Pick a vertex C in our original path which has an edge not contained in P. Then show there is a path P' from C to itself that shares no edges with P. (Hint: Use that the graph missing edges has all even degrees). Use complete sentences to explain.

(e) Put together P and P' to get an even long path. Why is this absurd?

(f) Conclude that our graph has an efficient path.