

## ENUMERATIVE COMBINATORICS

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Enumerative combinatorics is a field of mathematics that deals with counting the number of objects satisfying some combinatorial description.

### 1. CATALAN NUMBERS

1.1. **Exercise.** A *monotonic path* is a finite sequence of ‘up’ and ‘right’ steps of unit length. Count the number of monotonic paths from  $(0, 0)$  to  $(n, m)$ .

1.2. **Exercise.** Consider monotonic paths from  $(0, 0)$  to  $(n, n)$  which cross above the diagonal, i.e., those that enter the region  $y > x$ . Count the number of such paths by establishing a bijection with monotonic paths from  $(0, 0)$  to  $(n - 1, n + 1)$ .

1.3. **Exercise.** Count the number of monotonic paths from  $(0, 0)$  to  $(n, n)$  which do not cross above the diagonal, i.e., those that stay in the region  $y \leq x$ .

1.4. **Exercise.** Write the answer from the previous exercise as a fraction of a binomial coefficient. This is known as the *Catalan number*  $C_n$ .

1.5. **Exercise.** Let  $\nearrow = (1, 1)$  and  $\searrow = (1, -1)$  be vectors in  $\mathbb{Z}^2$ . Let

$$S = \left\{ (v_1, v_2, \dots, v_{2n}) \in \{\nearrow, \searrow\}^{2n} : \sum_{i=1}^{2n} v_i = (2n, 0) \right\}$$

be the collection of all sequences of length  $2n$  with the same number of  $\nearrow$  as  $\searrow$ . Count the number  $|S|$  of such sequences.

1.6. **Exercise.** Let  $B$  be the sequences  $(v_1, \dots, v_{2n}) \in S$  such that each partial sum  $s_t = \sum_{i=1}^t v_i$  has non-negative  $y$ -coordinate for  $t \in [2n] = \{1, 2, \dots, 2n\}$ . These are known as *ballot sequences*—a way for  $2n$  people to cast yes/no votes one at a time so that there are always at least as many “yes” as “no” votes, with a tie as the end result. Count the number  $|B|$  of ballot sequences.

1.7. **Exercise.** Prepend  $v_0 = \nearrow$  in front of every sequence in both  $S$  and  $B$ , and call the resulting sequences *augmented* and denote the sets  $\overline{S}$  and  $\overline{B}$ , respectively. For each augmented sequence  $(v_0, v_1, \dots, v_{2n}) \in \overline{S}$ , show that there is a unique  $r \in [0, 2n] = \{0, 1, \dots, 2n\}$  such that  $(v_{0+r}, v_{1+r}, \dots, v_{2n+r})$  is an augmented ballot sequence in  $\overline{B}$ , where the indices are read modulo  $2n + 1$ . Call the map  $f : \overline{S} \rightarrow \overline{B}$  that sends the augmented sequence to its corresponding augmented ballot sequence.

1.8. **Exercise.** Show that  $f$  is surjective and that the fibers under  $f$  are all of the same size, and calculate this size. That is, calculate  $|f^{-1}(b)|$  for  $b \in B$ .

1.9. **Exercise.** Establish a linear relationship between  $|B|$  and  $|S|$ .

1.10. **Exercise.** Conclude a second proof of the expression of the Catalan number in 1.4 *without* using subtraction, but using division instead.