PROOF WITHOUT WORDS

MATH CIRCLE (BEGINNERS) 05/06/2012

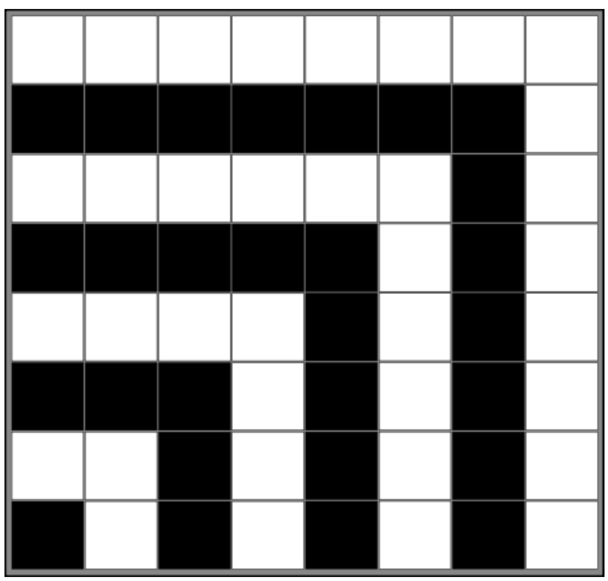
If you've been with us for a little while, you've already seen some examples of proofs without words.

Remember a proof is just an airtight argument that something must be true. If you can make that "argument" by showing a picture, then that picture counts as a proof! Here's an example.

Theorem: The sum of the first n odd integers is equal to n^2—in other words,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Proof:



□

(That square at the bottom right marks the end of the proof.)

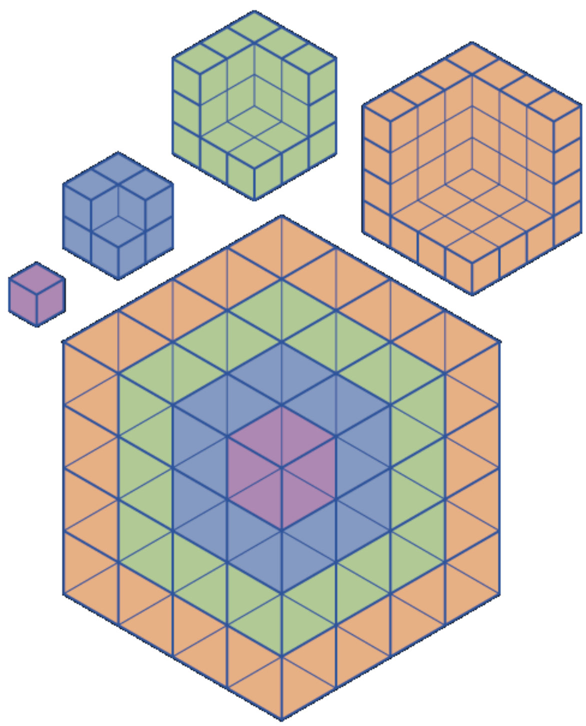
Now since this is a proof without words, it requires some interpretation by you, the proof reader.

(1) What part of the figure represents the odd number 5?

(2) This figure represents the sum of the first how many odd numbers?

(3) This particular image represents only the sum of a *specific number* of odd numbers. Does it still count as a proof of the general statement? Why or why not? (Hint: Remember what a proof is: it's a logical argument that completely convinces any skeptical reader of the truth of the claim.)

Another very similar example we saw recently was this:



(4) Write down, in words and/or math symbols, the statement for which the above image is a proof without words. (Hint: The numbers being added up are known as "hex numbers.")

(5) Give your own proof without words of the following theorem.

Theorem: The sum of the first n whole numbers is equal to half the product of n and $n+1$. In other words,

$$1+2+3+4+\cdots+n=\frac{1}{2}n(n+1).$$

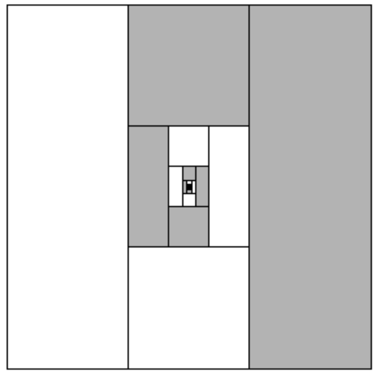
(6) Take an ordinary 8x8 chessboard and remove the lower left and upper right corner squares. Can you then tile the chessboard using only 2x1 dominoes with no gaps or overlap? (Hint: The proof does not need to be a proof without words.) (Hint2: what color are the removed squares?)

(7) Now suppose that instead of removing two opposite corners, you remove ANY TWO SQUARES OF OPPOSITE COLORS. Give a proof without words that it is always possible to tile the remaining squares using 2x1 dominoes.

Here is one way to do it: find a continuous path through all 64 squares, moving up left down or right, which meets up with itself to form a loop. The path should use every square exactly once. Since it only moves up down left and right, it will alternate square colors at every step. Explain how this proves that it's always possible to tile when we remove two squares of opposite colors.

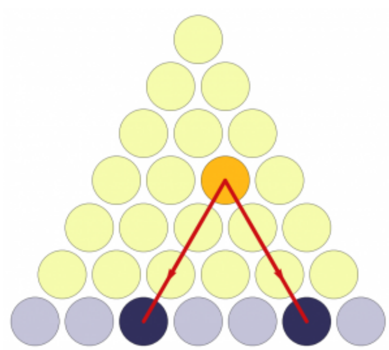
It may help a lot to try a smaller example first, say a 4x4 board.

(8) Here is a proof without words that demonstrates the value of a certain infinite sum (similar to ones that we saw earlier this year):



What is the infinite sum, and what is its value? (Hint: What fraction of the overall square does the largest shaded rectangle take up? The second largest shaded rectangle? The third largest? What is the pattern?)

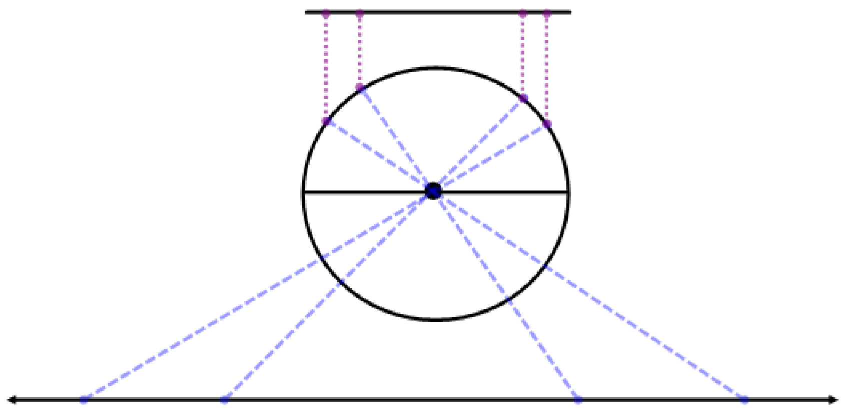
(9) The following picture is a proof without words of another interesting fact about the sum of the first n whole numbers, where in this case n happens to equal 6:



Can you figure out what is being proven? (Hint: It's a proof that the sum of the first n whole numbers is equal to... the number of ways to do what?) (Hint: The exact bold yellow circle, and the exact two bold blue circles, are not important... what's important is that we can do something similar for any of the yellow circles, or any two of the blue circles.)

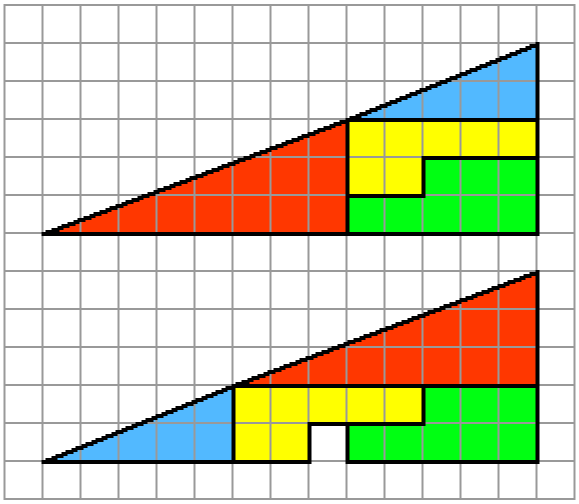
(10) Think of all the points on the number line.

Now think of just the points between -1 and 1, that is all points x such that $0 < x < 1$. Intuitively, there should be fewer of them, since they are strictly contained in the set of all real numbers.



What does this picture tell you?

(11) Proofs without words can be misleading if you're not careful. What does the following diagram prove? (What is the area of the upper triangle, and of the lower triangle? But...)



Can you explain what went wrong?

Images courtesy of

(1) Wikimedia Commons, http://en.wikipedia.org/wiki/File:Proofwithoutwords.svg

(2) Hexnet.org, http://hexnet.org/post/112

(3) Memtropy, http://memtropy.com/proofs-without-words/

(4) Math $\cap$ Programming, http://jeremykun.wordpress.com/tag/proofs-without-words/

(5) MathOverflow, http://mathoverflow.net/questions/8846/proofs-without-words