MODULAR ARITHMETIC I

MATH CIRCLE (INTERMEDIATE) 4/15/2012

Recall that integers a, b are called *congruent modulo* m (written $a \equiv b(\text{mod} m)$ if a and b have the same remainders when divided by m.

0) Prove that $a \equiv b(\text{mod} m)$ if any only if $a - b$ is divisible by m.

1) Prove the following:

a) If $a \equiv b(\text{mod} m)$ and $c \equiv d(\text{mod} m)$ then $a + c \equiv b + d(\text{mod} m)$.

b) If $a \equiv b(\text{mod} m)$ and $c \equiv d(\text{mod} m)$ then $a - c \equiv b - d(\text{mod} m)$.

2) Prove or disprove:

a) If $a \equiv b(\mathbf{mod} m)$ and $c \equiv d(\mathbf{mod} m)$ then $a \cdot c \equiv b \cdot d(\mathbf{mod} m)$.

b) If $ka \equiv kb(\mathbf{mod} m)$ then $a \equiv b(\mathbf{mod} m)$.

c) If $a \equiv b(\mathbf{mod} m)$ and n is a natural number, then $a^n \equiv b^n(\mathbf{mod} m)$.

3) Prove:

a) $30^{99} + 61^{100}$ is divisible by 31.

b) $43^{101} + 23^{101}$ is divisible by 66.

c) If n is odd, then $a^n + b^n$ is divisible by $a + b$.

4) a) Let s be a perfect square. Show that $s \equiv 0$ or $1 (\text{mod} 3)$. Similarly show $s \equiv 0$ or $1 (\text{mod} 4)$.

b) Let $k \neq 2$ be a product of the first several primes (i.e. $k = 2 \cdot 3$, or $k = 2 \cdot 3 \cdot 5$, etc.). Prove that both $k - 1$ and $k + 1$ are NOT perfect squares.

Notation: we will use the notation $\overline{a_1 a_2 \cdots a_n}$ to denote the number with digits $a_1, a_2, \ldots, a_n$. That is,

$$\overline{a_1 a_2 \cdots a_n} = a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \cdots + a_{n-1} \cdot 10 + a_n.$$

Note the tools above allow us to come up with some basic divisibility rules. Let $m = \overline{a_1 a_2 \cdots a_n}$.

k	m is divisible by k if
2	a_n is even.
3	$a_1 + a_2 + \cdots + a_{n-1} + a_n$ is divisible by 3.
4	$\overline{a_{n-1} a_n}$ is divisible by 4.
5	a_n is 0 or 5.
6	m is divisible by 2 and 3.
9	$a_1 + a_2 + \cdots + a_{n-1} + a_n$ is divisible by 9.

5) Prove the above divisibility rules.

6) Prove that if you reverse the order of the digits in any natural number and subtract the result from the initial number, then the difference is divisible by 9.

7) Find all natural numbers whicih become 9 times greater if you insert a 0 between their ones digit and their tens digit.

8) Does there exist a natural number n such that $n^2 + n + 1$ is divisible by 1955?

9) Find all 3 digit numbers, any power of which ends with three digits forming the original number.

Challenge 1) Prove that there exist infinitely many natural numbers that cannot be represented as a sum of cubes.

Challenge 2) Prove that the number $\overline{a_1 a_2 \cdots a_n a_n a_{n-1} \cdots a_1}$ is composite.

Challenge 3) Let n be a natural number such that $n+1$ is divisible by 24. Prove that the sum of all the divisors of n is also divisible by 24.

Problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
- Previous UCLA Math Circle notes