Homogeneous Coordinates

The *homogeneous coordinates* of a point $(X, Y) \in \mathbf{R}^2$ are all the real ordered triples (Xz, Yz, z) with $z \neq 0$, i.e. all triples (x, y, z) such that $x/z = X$ and $y/z = Y$. If we take (X, Y) to be coordinates in the plane $z = 1$, then these are the coordinates of points on the line from the origin to (X, Y). We denote homogeneous coordinates by $(x : y : z)$.

Thus homogeneous coordinates give a one-to-one correspondence between points $(X, Y) \in \mathbf{R}^2$ and nonhorizontal lines through the origin in $\mathbf{R}^3$. The horizonatal lines, which have homogeneous coordinates of the form $(x, y, 0)$, naturally correspond to points at infinity.

1. Which of the following homogeneous coordinates represent the same points? Which represent points at infinity?

   $(-1 : -1 : -1)$ $(2 : 3 : 5)$ $(0 : 0 : 1)$ $(-1 : 0 : 0)$ $(2 : -3 : 0)$ $(2 : 2 : 2)$
   $(4 : -6 : 0)$ $(1 : \frac{3}{2} : \frac{5}{2})$ $(6 : 6 : 6)$ $(0 : 1 : 3)$ $(0 : 0 : 3)$ $(10000 : 10000 : 10000)$

2. We say that $p(x, y, z)$ is a *homogeneous polynomial* of degree m if

   - p is a sum of terms of the corm $a_{jk\ell}x^j y^k z^\ell$, where $j + k + \ell = m$

   OR

   - For any t, $p(tx, ty, tz) = t^m p(x, y, z)$.

   (See problem (2b) below.)

   We make the analogous definition in one, two, or any other number of variables.

   (a) Identify the homogeneous polynomials and give their degree:
   xyz x^2-y^2 $2x+2y-z$ $x^2+3y^2+5z^2$ $xy-y+z^3$ $xy+yz+3xz$
   $xy^3 - x^4 + 3xy^2z - 6xz^2$ $y - x^2$ $3xy + 50y^2 - z$ $x^{10} + y^5z^5 - 2x^3z^7$

   (b) Prove that the two definitions above are equivalent.

3. An *algebraic curve* is a curve defined by an equation $p(X,Y)=0$, where p is a polynomial in X,Y. Examples:

If p is a polynomial of degree n, we can write the equation $p(X,Y)=0$ as $p(\frac{x}{z},\frac{y}{z})=0$ when $z\neq 0$, and then extend to all values of z by multiplying through by z^n:

$$\bar{p}(x,y,z)=0, \quad \text{where} \quad \bar{p}(x,y,z)=z^n p\left(\frac{x}{z},\frac{y}{z}\right).$$

Example: The equation $X^3+XY-Y=0$ becomes $\left(\frac{x}{z}\right)^3+\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)-\frac{y}{z}=0$ becomes $x^3+xyz-yz^2=0$.

Having done this, we can include *points at infinity* which are solutions of the extended polynomial equation satisfying $z=0$.

(a) For each curve, find how many points at infinity there are for $p(X,Y)=0$

$X+2Y \qquad X^2-Y^2 \qquad X^2+Y^2 \qquad Y-X^2$

(b) How many points of intersection (including points at infinity) are there between: i. $y=0$ and $y=x^2+1$ ii. $x+y=0$ and $x-y=0$ iii. $xy=1$ and $y=-x$ iv. $2x+y=0$ and $2x+y=1$