**LAMC Junior Circle** **March 18, 2012**

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# The Sierpinski triangle and the Hanoi Tower Puzzle.

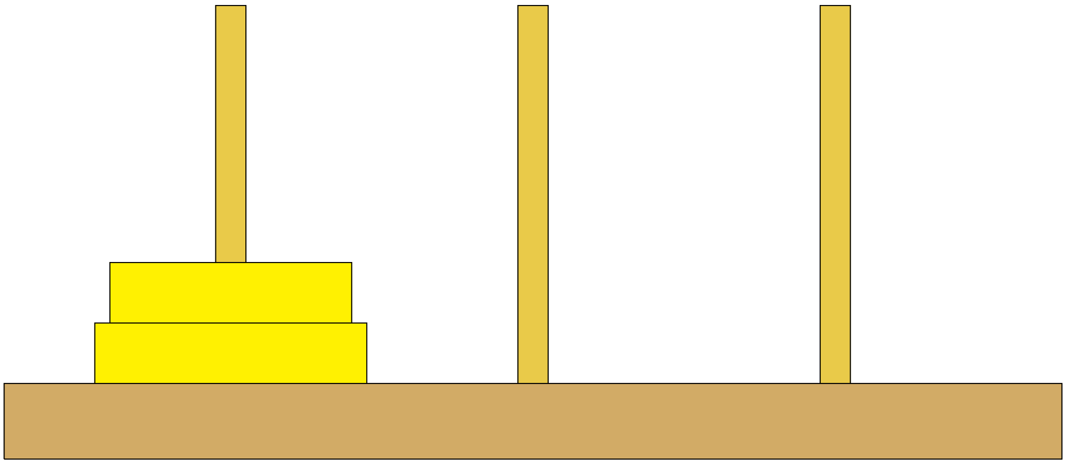


## The Hanoi Towers states.

Recall that the Hanoi Tower puzzle has three rods, $A$, $B$, and $C$, and $n$ disks. At the beginning, all the disks are on rod $A$. The goal is to move the disks to rod $C$ in such a way that

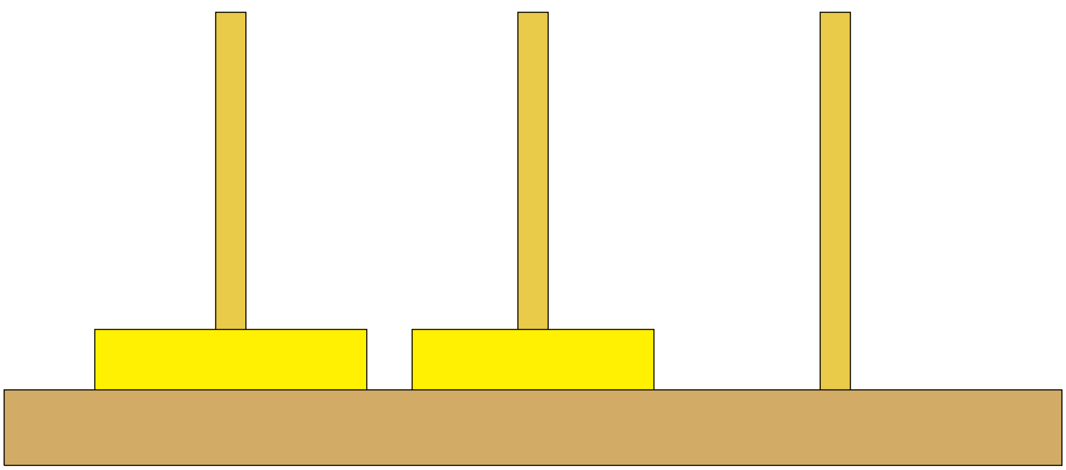
- only one disk is moved at a time;
- a larger disk cannot be put on top of a smaller one.

Let us call a *state* of the puzzle a complete description of all the disks' positions. For example, let us encode the original position of the puzzle with two disks as $AA$. The first letter describes the location of the first disk. It is located on rod $A$, so the first letter is $A$. The second disk is also located on rod $A$, so the second letter is $A$ as well.



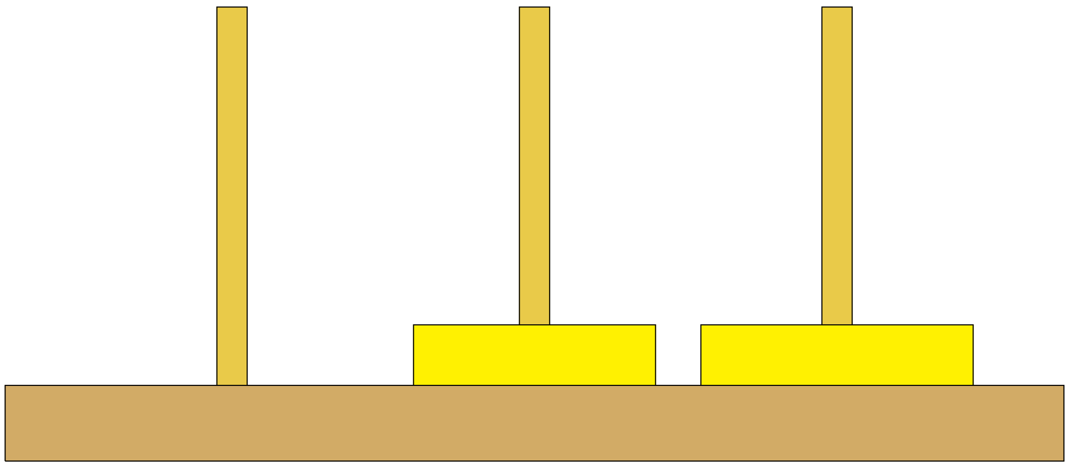
$AA$

The first move to solve the puzzle is $1AB$. The resulting state is $BA$. This means that the first disk is now on rod $B$ while the second is still on rod $A$.



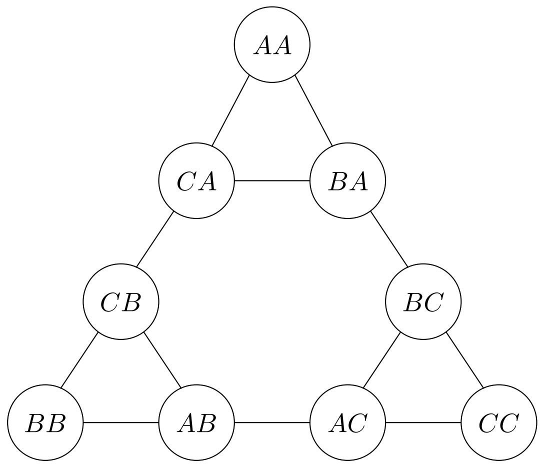
$BA$

**Problem 1** *The next move to solve the puzzle is* $2AC$. *Write down the resulting state.*



**Problem 2** *What is the next move to solve the puzzle? What is the corresponding state?*

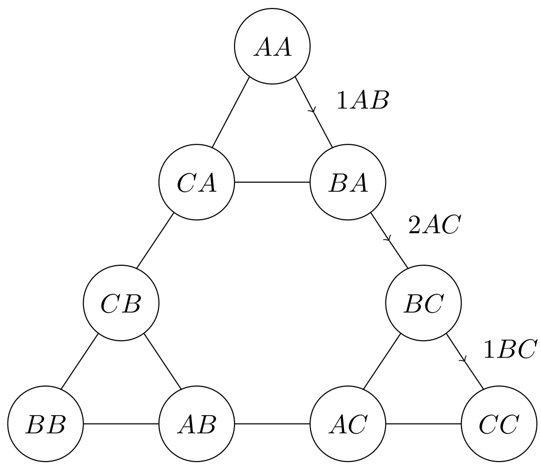
A *graph* is a set of nodes, called vertices, and a set of links connecting them, called edges. Graphs are of great importance in mathematics. For example, we can represent the states of the Hanoi Tower puzzle with two disks by the following graph.



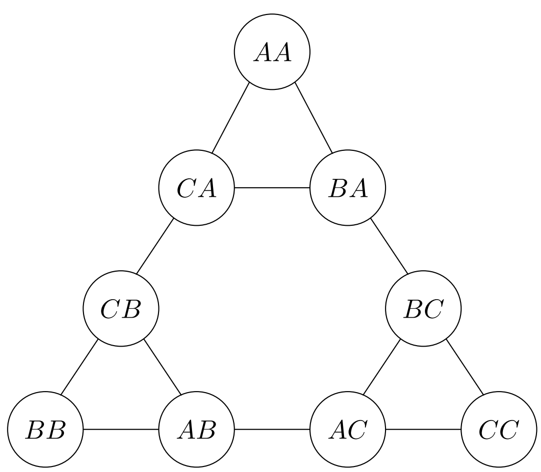

**Problem 3** *Describe in your own words the meaning of the state* $BB$*.*

**Problem 4** *Write down the move that turns the state* $CB$ *into the state* $CA$*.*

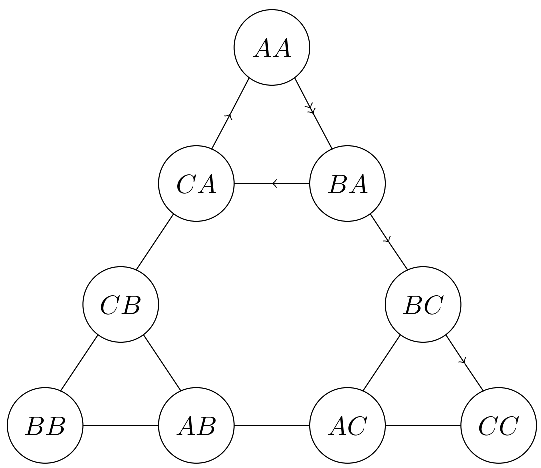
Note that we can describe the moves graphically by adding arrows to the corresponding edges of the graph. For example, the sequence of moves that solves the puzzle with two disks forms the following path.




**Problem 5** *Use arrows to mark the path in the graph corresponding to the optimal algorithm that moves two disks from rod B to rod A. Write down the moves next to the corresponding edges.*



Suppose that playing with the puzzle, we repeat a state. This means that we have used the corresponding configuration of the disks before and now get back to it, by mistake or on purpose. The corresponding path in the graph has a *self-intersection*. For example, the following self-intersecting path




corresponds to the algorithm

$$1AB\ 1AC\ 1AA\ 1AB\ 2AC\ 1BC$$

that solves the puzzle, but is not optimal.

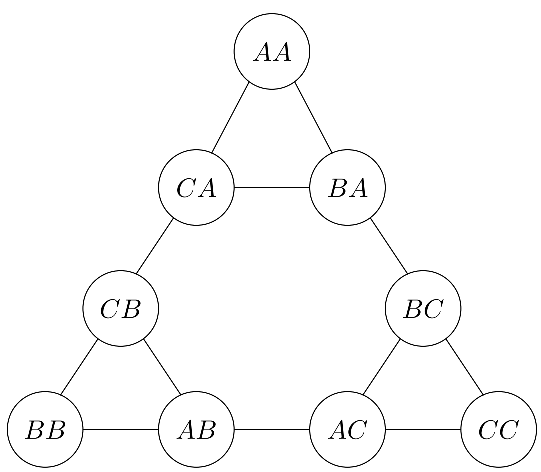
In the distant future, a human space traveler is captured on a remote planet by some evil aliens. It is their custom to make a captive play her/his favorite game. Once the game is finished, win or lose, the prisoner is executed.

This is the plot of the sci-fi story "Now Inhale", by Eric Russel. In the story, the prisoner chooses to play the Hanoi Tower puzzle with 64 disks, postponing his execution by more than five hundred billion years even in the case of the shortest winning algorithm.

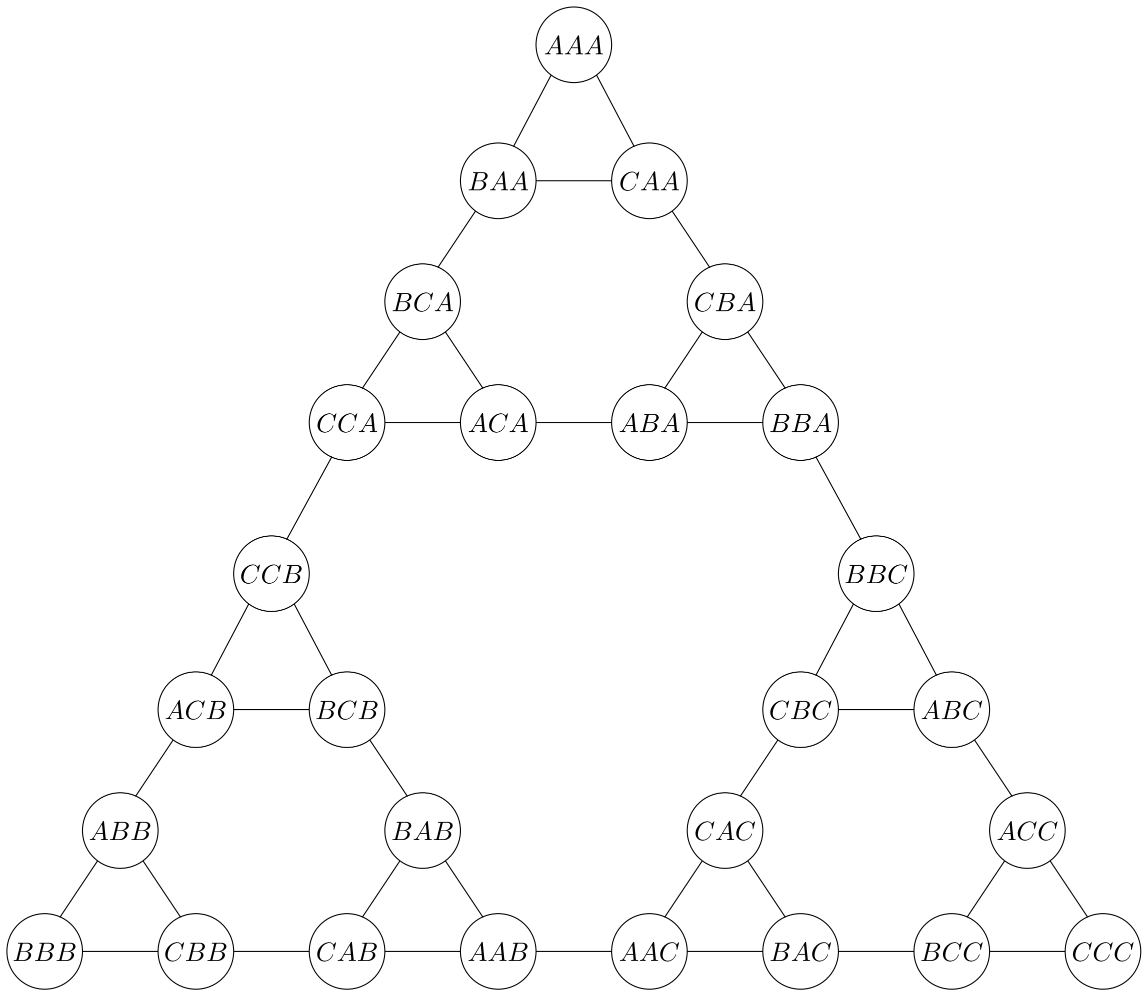
Let us prepare ourselves for a similar misfortune. Suppose that the evil aliens grew smarter and may limit the number of disks. The rescue ship can take some time to arrive, so you want to use not the shortest, but the longest algorithm to move all the disks from rod $A$ to rod $C$. However, do not repeat the states – the aliens may suspect you of cheating and execute right away!

The above boils down to the following problem. Find the longest way to move the disks from rod $A$ to rod $C$ without repeating the states.

**Problem 6** *Use the graph below to solve the problem for two disks. What is the length of the longest algorithm?*



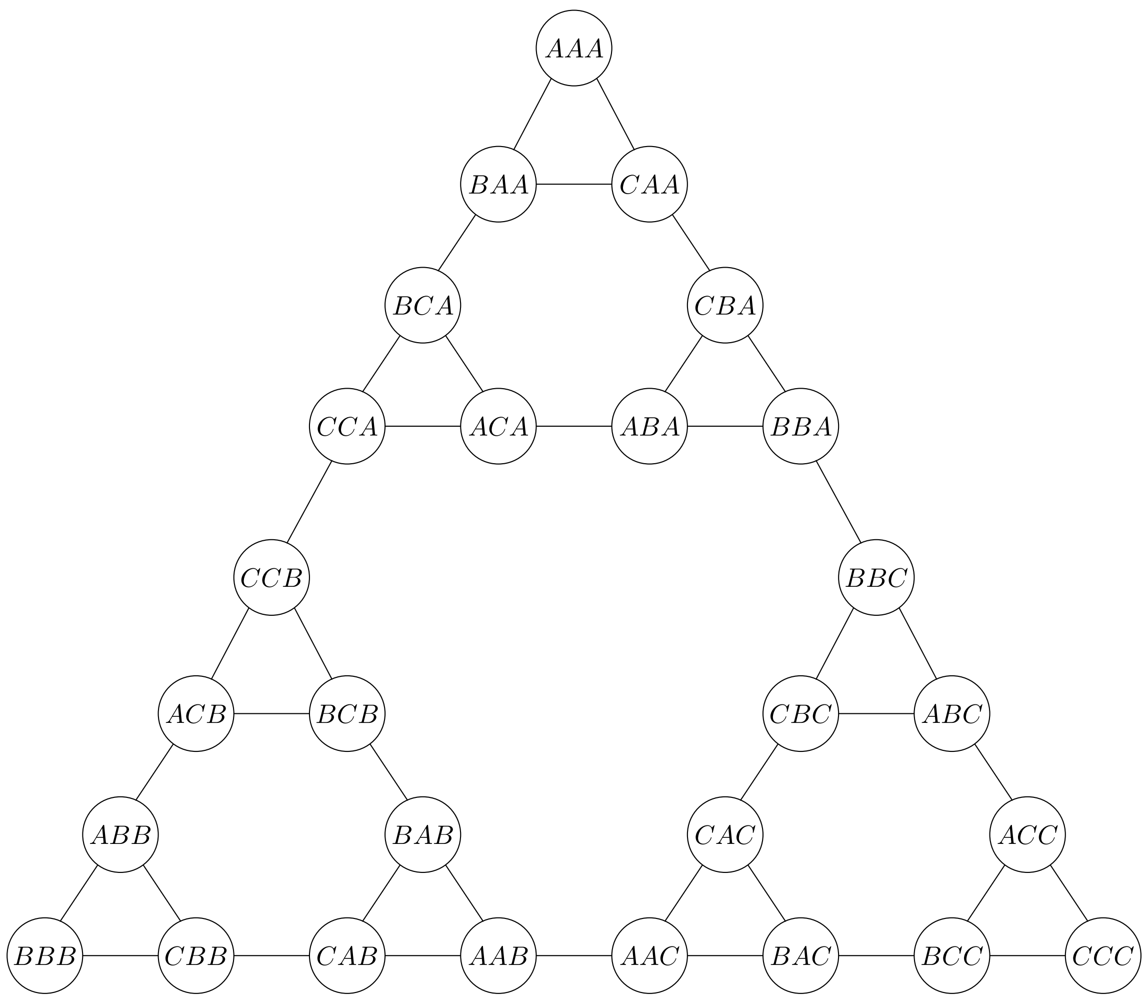
Let us consider the graph representing the states of the Hanoi Tower puzzle with three disks.



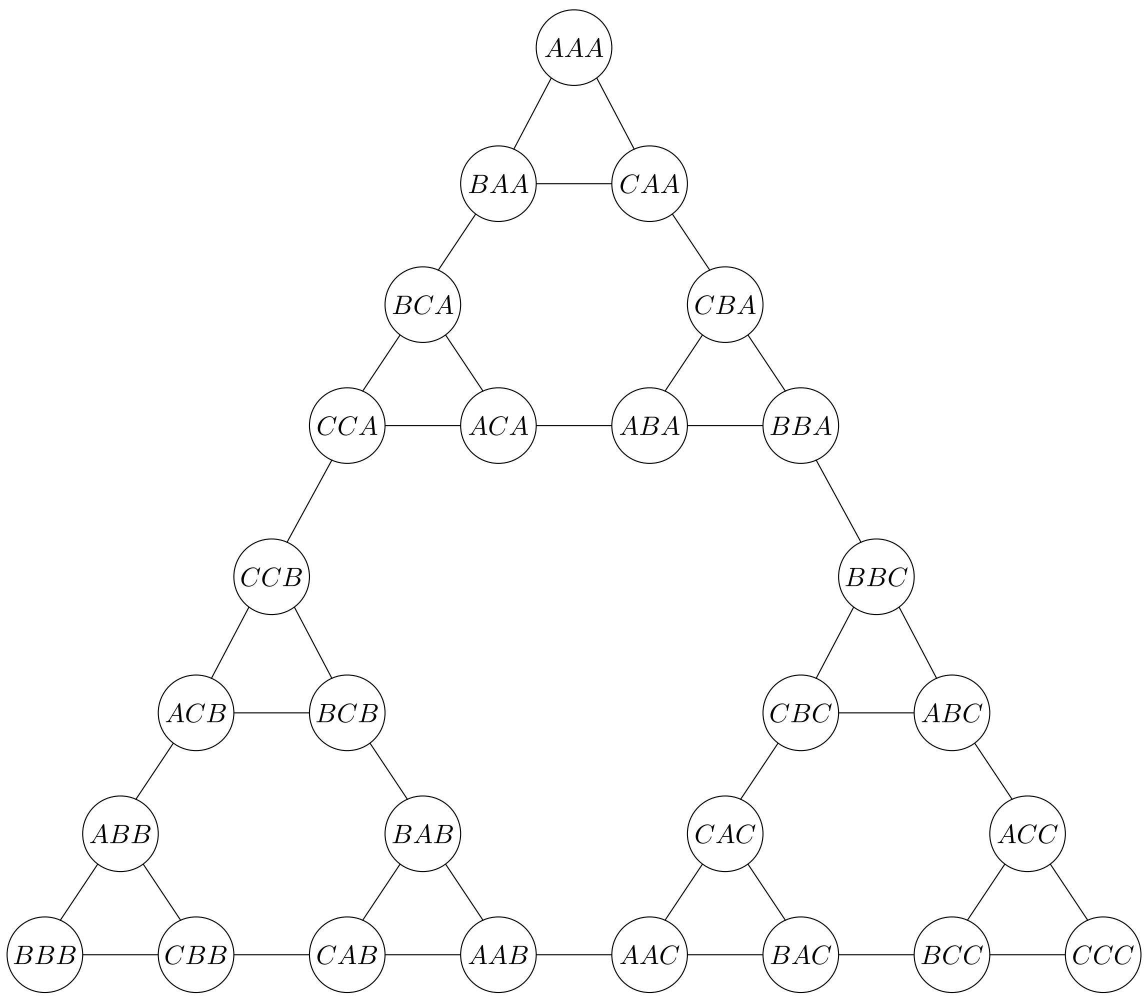

**Problem 7** *Describe in your own words the meaning of the state $ABB$.*

**Problem 8** *Write down the shortest algorithm turning the state $CBC$ into the state $AAB$.*

*Mark with arrows the corresponding path on the graph below.*

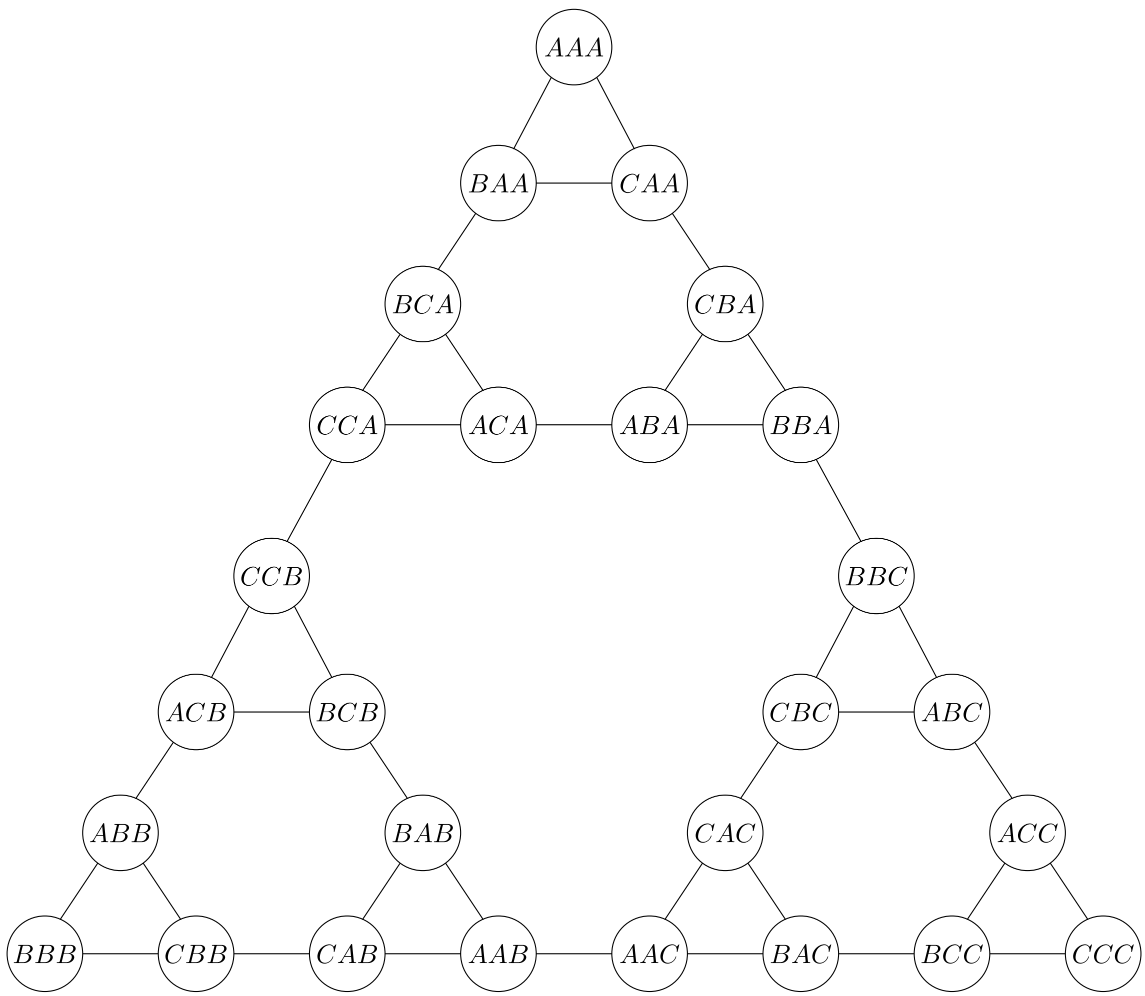


**Problem 9** *Mark with arrows the shortest path from the original state $AAA$ to the solution state $CCC$ on the graph below. Write down the moves next to the corresponding arrows.*

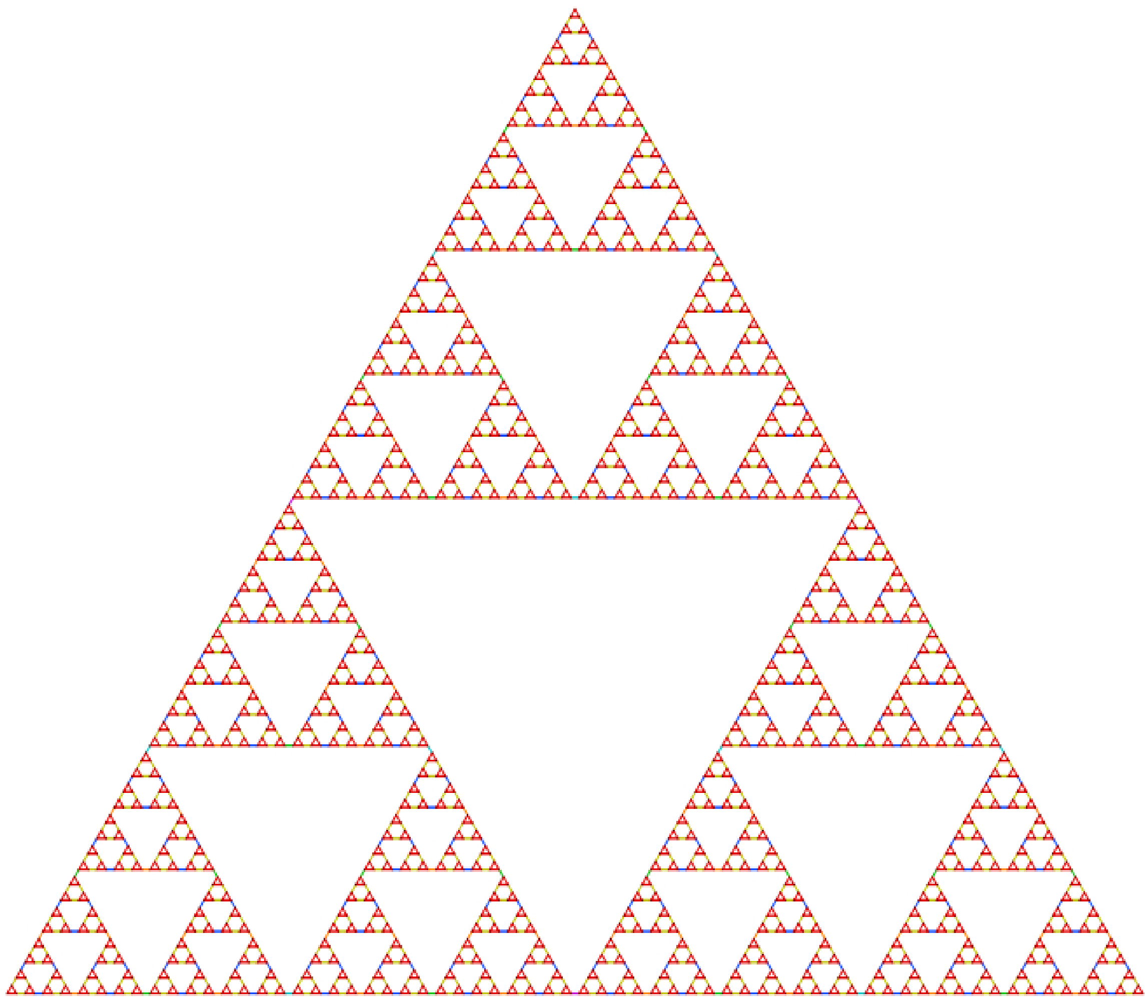


Let us continue our preparations for meeting the evil aliens from Problem 6.

**Problem 10** *Mark with arrows the longest path without self-intersections joining the original state $AAA$ with the solution state $CCC$ on the graph below. How long is the corresponding algorithm?*



The following graph represents the states of the Hanoi Tower puzzle with seven disks. It looks very similar to the sixth approximation of the Sierpinski triangle, $ST_6$.



States of the Hanoi Tower puzzle with seven disks.

In fact, the Sierpinski triangle, $ST$, can be interpreted as the states graph of the Hanoi Tower puzzle with infinitely many disks.

**Problem 11** *Draw the Sierpinski triangle approximation similar to the graph representing the states of the Hanoi Tower puzzle with three disks.*

**Problem 12** *Solve the following cryptarithm.*

$$\begin{array}{rcccccc} & N & U & M & B & E & R \\ + & N & U & M & B & E & R \\ \hline & P & U & Z & Z & L & E \end{array}$$