# INDUCTION III AND PROOF BY CONTRADICTION

MATH CIRCLE (INTERMEDIATE) 3/4/2012

1) Show that $n$ straight lines (such that no two are parallel and no three meet at the same point) divide a plane into $1 + (1 + 2 + \cdots + n) = 1 + \frac{n(n+1)}{2}$ parts.

2) A bank has an unlimited supply of $3$-peso and $5$-peso notes. Prove that it can pay any number of pesos greater than $7$. (Hint: prove the result for $8, 9, 10$ directly, and then use MMSI (strong induction).)

3) Suppose we have a football league with $n$ teams. Every team plays each other exactly once (and there are no ties). Prove that there is a way to number $1, 2, \ldots, n$ so that team $1$ beat team $2$, $2$ beat $3$, etc.

4) $2n$ dots are placed around the outside of the circle. $n$ of them are colored red and the remaining $n$ are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

Another useful method of proof is called Proof by Contradiction (in latin: "Reductio ad absurdum").

Suppose we are trying to prove a statement $P$. Instead of directly trying to prove $P$, we instead assume that $P$ is false and reduce this to a contradiction or an absurdity.

In other words, we are showing that $P$ cannot be false, thus it must be true!

Think of an alibi: Suppose during math circle something is stolen from my apartment, and I accuse you of being the thief. You would say: "I am not the thief (statement $P$), because if I was the thief ($P$ is false), then I would have been here at math circle and at your apartment at the same time (a contradiction!)".

5) A number is "interesting" if it has a noteworthy property. Prove that there are infinitely many interesting numbers.

6) Prove that there are infinitely many primes. (Hint: Suppose there are only finitely many primes. Find a prime bigger than all of them.)

7) Suppose $a$ and $b$ are integers. If $4|(a^2 + b^2)$, then $a$ and $b$ are not both odd.

Challenge 1) Prove that $\sqrt{2}$ is an irrational number.

Challenge 2) Prove that among any $2^{n+1}$natural numbers there are $2^n$ numbers whose sum is divisible by $2^n$. (Hint: You somehow want to end up with $3$ subsets of size $2^{n-1}$ that are divisible by $2^{n-1}$.)

Problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
- Previous UCLA Math Circle notes