INDUCTION II

MATH CIRCLE (INTERMEDIATE) 2/26/2012

Recall the Method of Mathematical Induction (MMI) is a way to show that a property $P(n)$ holds for all $n = 1, 2, 3, \ldots$. A proof using MMI consists of two steps:

Basis: Show that $P(1)$ holds.

Induction: Assume $P(k)$ holds, and show that $P(k+1)$ holds.

Then, if the basis and induction statements hold, MMI tells us that $P(n)$ holds for all $n = 1, 2, 3, \ldots$.

1) Show that for all $n \geq 1$, $3^n > n^2$.

Basis Step:

Induction Step:

We assume that ____.

We want to show:

2) Show that $8^n - 1$ is always divisible by 7. Recall we write this as $7|8^n - 1$.

Basis Step:

Induction Step:

We assume that ____.

We want to show:

3) Show that $9|4^n + 15n - 1$ for all $n \geq 1$.

Basis Step:

Induction Step:

We assume that ____.

We want to show:

An expanded Method of Mathematical Induction (MMI) is a way to show that a property $P(n)$ holds for all $n \geq k_0$ for some "starting point" k_0. A proof using MMI consists of two steps:

Basis: Show that $P(k_0)$ holds.

Induction: Assume $P(k)$ holds, and show that $P(k+1)$ holds.

Then, if the basis and induction statements hold, MMI tells us that $P(n)$ holds for all $n = k_0, k_0 + 1, k_0 + 2, \ldots$.

4) Prove that $n! > 2^n$ for all $n \geq 4$.

Basis Step:

Induction Step:

We assume that ____.

We want to show:

More generally, the Method of Mathematical Strong Induction (MMSI) is a way to show that a property $P(n)$ holds for all $n \geq k_0$. A proof using MMSI consists of two steps:

Basis: Show that $P(k_0)$ holds.

Induction: Assume $P(k_0), P(k_0+1), \ldots, P(k)$, and show that $P(k+1)$ holds.

Then, if the basis and induction statements hold, MMSI tells us that $P(n)$ holds for all $n = k_0, k_0+1, k_0+2, \ldots$.

5) Suppose we define a sequence a_n, such that $a_1 = 1, a_2 = 2, a_3 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Prove that $a_n < 2^n$ for all $n \geq 4$.

Basis Step:

Induction Step:

6) Prove that for any $n \geq 2$, n can be written as a product of primes.

Basis Step:

Induction Step:

7) What is wrong with the following proof (by MMI) that all horses have the same color:

Let $P(n)$ be the statement that in any collection of n horses, all the horses have the same color.

Basis Step: $P(1)$ is true because if there is only one horse, there is only one color.

Induction Step: Assume that in any collection of n horses, all the horses have the same color.

Suppose we have a collection of $n+1$ horses. Number the horses $1, 2, \ldots, n+1$. Consider the collections $1, 2, \ldots, n$ and $2, 3, \ldots, n+1$. By induction, horses $1, 2, \ldots, n$ have the same color, as do horses $2, 3, \ldots, n+1$. Therefore, all $n+1$ horses have the same color.

Thus, MMI tells us that $P(n)$ is true for all n, hence all horses have the same color.

Problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
- Previous UCLA Math Circle notes