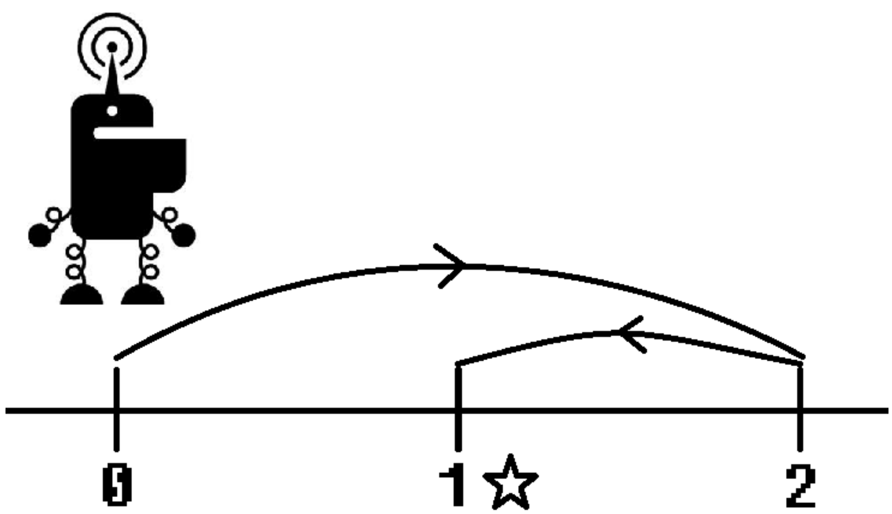
# HERE'S LOOKING AT EUCLID 2: EUCLID'S REVENGE

MATH CIRCLE (BEGINNERS) 02/19/2012

Remember your job at the **HopBot Testing Facility**, home of the world's premier hopping robot? A **HopBot-X** will start at 0 on the number line, and repeatedly hop X spaces. So for instance, a HopBot-3 will hop: 0, 3, 6, 9, 12, 15, etc.

**BAD NEWS!** Many of your HopBots have been infected with a computer virus that makes them behave very strangely. An infected HopBot can hop some fixed number of spaces when hopping to the left, but a *different* number of spaces when hopping to the right. You need to get each virus-stricken HopBot back to its own recharging station in order to repair it.

For example, you have one sick HopBot which can hop 1 unit leftward, or 2 units rightward. If it starts at 0 on the number line, and its recharging station is at 1 (marked with a star), it can get there by hopping 2 units rightward and 1 unit leftward:




We will call an infected HopBot that hops $A$ spaces leftward and $B$ spaces rightward, a Hopbot-($\leftarrow A, B \rightarrow$), so the HopBot above is a Hopbot-($\leftarrow 1, 2 \rightarrow$), since it can go 1 space left or 2 spaces right.

(1) Can the Hopbot-($\leftarrow 1, 2 \rightarrow$) reach *any* whole number on the number line? Why or why not?

(2) You have an infected Hopbot-($\leftarrow 7, 3 \rightarrow$) which is currently at 0 on the number line. Can you return it to its recharging station at 1? If so, give a sequence of hops that works (you can write it like Left-Left-Left-Left-Right-Left, or LLLLRL, or 4L,1R,1L). If not, explain why it can't be done.

(3) You have an infected Hopbot-($\leftarrow 9, 22 \rightarrow$) which is currently at 0 on the number line. Can you return it to its recharging station at 1? If so, give the sequence of hops, otherwise explain why it can't be done.

(4) You have an infected Hopbot-($\leftarrow 21, 30 \rightarrow$) which is currently at 0 on the number line. Can you return it to its recharging station at 1? If so, give the sequence of hops, otherwise explain why it can't be done.

(5) You have an infected Hopbot-($\leftarrow 10, 24 \rightarrow$) which is currently at 0 on the number line. Can you return it to its recharging station at 6? If so, give the sequence of hops, otherwise explain why it can't be done.

(6) You have an infected Hopbot-($\leftarrow 41, 37 \rightarrow$) which is currently at 0 on the number line. Can you return it to its recharging station at 1? If so, give the sequence of hops, otherwise explain why it can't be done.

(7) For your answers in problems (2)-(6), does it matter the precise order the infected HopBot makes its hops, or just the total number of left and right hops?

(8) For each HopBot in problems (2)-(6), describe all possible numbers that HopBot can get to, starting from 0.

(9) What is the relationship between the numbers a Hopbot-($\leftarrow A, B \rightarrow$) can reach, and the numbers $A$ and $B$? (Hint: There is a very specific answer you can give here!)

The **Extended Euclidean Algorithm** is a way to solve equations of the form

$$ax + by = c,$$

where $a, b, c$ are known integers, and you're looking for integer values of $x$ and $y$ that make the equation true (assuming they exist!).

Remember the ordinary Euclidean algorithm? Here's a reminder of the process, along with a specific example to refresh your memory:

$$\begin{array}{rcccll}
a & = & q_1 b & + & r_1 & (0 < r_1 < b) \\
b & = & q_2 r_1 & + & r_2 & (0 < r_2 < r_1) \\
r_1 & = & q_3 r_2 & + & r_3 & (0 < r_3 < r_2) \\
r_2 & = & q_4 r_3 & + & r_4 & (0 < r_4 < r_3) \\
 & & \vdots & & & \\
r_{n-2} & = & q_n r_{n-1} & + & r_n & (0 < r_n < r_{n-1}) \\
r_{n-1} & = & q_{n+1} r_n & + & 0 &
\end{array}$$

Example: Find $\gcd(76, 32)$ (so in this case $a = 76$, $b = 32$):

$$\begin{array}{rcccc}
76 & = & 2 \cdot 32 & + & 12 \\
32 & = & 2 \cdot 12 & + & 8 \\
12 & = & 1 \cdot 8 & + & 4 \\
8 & = & 2 \cdot 4 & + & 0
\end{array}$$

Now that we've done that, we can use the **Extended** Euclidean Algorithm to find $x$ and $y$ such that $ax + by = gcd(a, b)$. In this particular case: $76x + 32y = 4$. The way to do it is to use the results from the ordinary Euclidean Algorithm, but to work backwards from the next-to-last line. To make things easier on us, let's start by rewriting the equations in an equivalent way, namely with the remainders by themselves on the left side:

$$\begin{array}{rccccccccccl}
76 & = & 2 \cdot 32 & + & 12 & \rightarrow & 12 & = & 76 & + & (-2) \cdot 32 & (\#1) \\
32 & = & 2 \cdot 12 & + & 8 & \rightarrow & 8 & = & 32 & + & (-2) \cdot 12 & (\#2) \\
12 & = & 1 \cdot 8 & + & 4 & \rightarrow & 4 & = & 12 & + & (-1) \cdot 8 & (\#3) \\
8 & = & 2 \cdot 4 & + & 0 & & & & & & &
\end{array}$$

Since equation #2 tells us that $8 = 32 + (-2) \cdot 12$, we can substitute $32 + (-2) \cdot 12$ into equation #3 in place of $8$. So equation #3 becomes

$$4 = 12 + (-1) \cdot (32 + (-2) \cdot 12)$$

Combining the 12's and 32's gives:

$$4 = 12 + (-1) \cdot 32 + 2 \cdot 12$$

$$4 = (-1) \cdot 32 + 3 \cdot 12 \qquad (\#4)$$

I called this new equation Equation #4. But look: Equation #1 tells us that $12 = 76 + (-2) \cdot 32$, so we can substitute $76 + (-2) \cdot 32$ into equation #4 now in place of $12$. Excellent! Equation #4 becomes

$$4 = (-1) \cdot 32 + 3 \cdot (76 + (-2) \cdot 32)$$
$$4 = (-1) \cdot 32 + 3 \cdot 76 + (-6) \cdot 32$$

$$4 = 3 \cdot 76 + (-7) \cdot 32$$

And we are done! This tells us that to solve $76x + 32y = 4$, we can take $x = 3$, $y = -7$. Hooray!!! See if you can use the Extended Euclidean Algorithm to solve the following equations:

(10) $11x + 7y = 1$

**(11)** $25x + 35y = 5$

**(12)** $16x + 7y = 3$
(Hint: First solve $16x + 7y = 1$. Then what?)