Mohr-Mascheroni Theorem

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A ruler-and-compass construction is a sequence of basic steps of the following three types:

1. **Intersection of two lines**: Given four points $A \neq B$,$C \neq D$, find the intersection point E of the line passing through A and B with the line passing through C and D.

2. **Intersecting line with circle**: Given five points $A \neq B$,C,$D \neq E$, find the two points F,G of intersection of the line passing through A and B with the circle centered at C with radius equal the distance $|DE|$.

3. **Intersection of two circles**: Given six points $A \neq B$, $C \neq D$,$E \neq F$ fine the two points G and H of intersection of the circle centered at A and radius the distance $|CD|$ with the circle centered at B with radius the distance $|EF|$.

We only permit to do constructions 2),3) when there are at least two intersection points. If there is only one point (tangential case), then this point can be found by a sequence of basic steps (how?).

Theorem 1 (Mohr-Mascheroni) *Every construction that can be done with ruler and compass can be done with compass alone.*

The theorem says that that points found in steps 1 and 2 can be found by a sequence of constructions using step 3 alone. This worksheet develops a proof of this theorem, following Norbert Hungerbühler in Amer. Math. Monthly 101 (1994). For the tasks below, use the ruler only for steps that you have already established to be possible by compass only.

1. Find the reflection point of a point C across a line through two given points A and B.
2. Find the intersection points of a line AB with a circle around a point C with radius $|DE|$, provided C itself is not on the line.
3. Given A, B, find C such that B is the midpoint of A and C.
4. Given points A and B, find the midpoint point C. (Hint: one can proceed using several isosceles triangles of sidelength ratio $1 : 2$)
5. Given a line AB and a point C outside the line, find the intersection of the line AB with the perpendicular to AB through C.
6. Prove Euler's theorem: Assume A, B, C, D are four points on a circle with $A \neq B$ and $C \neq D$, and assume E is the intersection point of the lines AB and CD. Prove that

   $$|AE| \times |BE| = |CE| \times |DE|$$

   This theorem is called theorem of chords if E is inside the circle and theorem of secants if E is outside the circle. Hint: it is enough to prove this when one line passes through the center of the circle (why?). Draw a perpendicular from the center of the circle to the other line and consider the Pythagorean theorem for two right triangles
7. Given a line AB and a point on the line C and a radius $|DE|$, construct two points with distance $|CF|$ where F is one of the intersection points of the line with the circle. (Use Euler's theorem for the given circle and a larger circle, note that we are already able to intersect lines with circles provided the lines do not pass through the center of the circle.
8. Construct the point F itself in the previous task.
9. Given two lines AB and CD, let E be the intersection of AB with the perpendicular to AB through C and let F be the intersection of CD with the perpendicular to CD through E. Use Euler's theorem to find two points of distance $|FG|$, where G is the intersection point of the lines AB and CD. Use this to construct the point G.