LAMC Early Elementary Circle October 23, 2011

Oleg Gleizer

How to break a number into parts?

Part 1

A common kindergarten and first grade problem is to break a positive integral number into a sum of two. In this lesson, we shall learn to break a positive integer n into a sum of positive integers in all possible ways, from one part, $n = n$, to n parts, $n = \underbrace{1 + \ldots + 1}_{n \text{ times}}$.

To make sure that every student is familiar with the word "story" as in the "three-story house", let us solve the following problem.

Problem 1 *Once upon a time on the tenth floor of a ten-story apartment building there lived a little boy who very much liked riding in the elevator. It was a very safe elevator, so the boy's parents didn't mind him taking rides unaccompanied by adults. Going down, the boy enjoyed it all the way to the first floor. On his way up however, he only took the elevator to the third floor and then walked seven stories up. A few months later, the boy started going in the elevator up to the fourth level instead of the third. Some more time has passed and he began taking the elevator up to the fifth floor. Can you explain the boys behavior?*

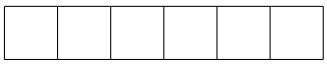
Example 1 *Represent 6 as a sum of smaller positive numbers. We shall approach the solution graphically. Let us call a square an "apartment". Let us draw a house built of 6 apartments in such a way that*

1. an upper floor cannot have more apartments than the lower floor and there shouldn't be gaps between the apartments (otherwise the house will collapse);

2. the house wall facing the street must be built vertical (according to the city regulations); and

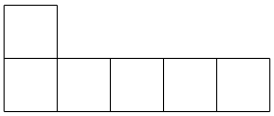
3. the street is always on the left.

First, here comes the one-story house. This is the partition of the number 6 into 1 piece.

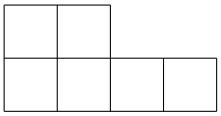


$6=6$

Then the two-story houses and the corresponding two-parts partitions.



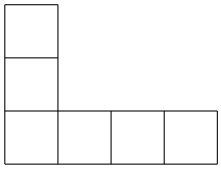
$6=5+1$



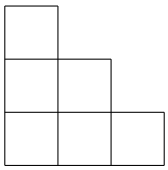
$6=4+2$

$6=3+3$

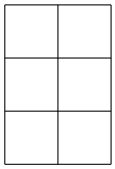
Then the time comes for the three-story houses.



$6=4+1+1$

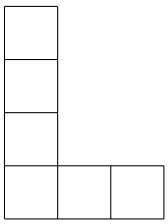


$6=3+2+1$

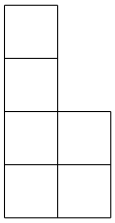


$6=2+2+2$

Here come the four-story houses and the corresponding partitions.

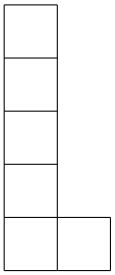


$6=3+1+1+1$



$6=2+2+1+1$

There is only one way to build a five-story building (why?).



$6=2+1+1+1+1$

And finally here comes the six-story tower.

$6=1+1+1+1+1+1$

Problem 2 *Draw all the possible houses with 4 apartments. Sum up the number of the apartments on each floor next to the corresponding picture as above.*

Problem 3 *Find the value of 2+3 by drawing a two-story house, 3 apartments on the first floor, 2 – on the second, and counting the total number of the apartments.*

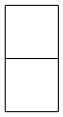
Problem 4 *Find the value of 7-3 by drawing a two-story house with seven apartments, 3 of them on the second floor, and by counting the number of the apartments on the first floor.*

Problem 5 *Using the above approach, find*

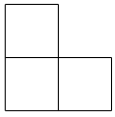
$$7\text{-}4=$$

Definition 1 *A (positive) integer is called even, if it can be represented by a two-story house with an equal number of apartments on each floor. An integer that is not even is called odd.*

1 is obviously not even. It cannot be represented by a two-story house, because we only have one apartment. So, 1 is an odd number. 2 is even.

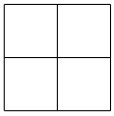


3 is odd.

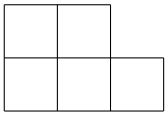


Problem 6 *Is there a way to represent 3 as a two story-house with an equal number of apartments on each floor? Why or why not?*

4 is even.

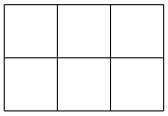


5 is odd.



It's time to notice the pattern: an odd number is obtained from the previous even number by adding an extra apartment to the first floor.

6 is even.



Observe that an even number is obtained from the previous odd number by adding an extra apartment on the top of the only first floor apartment which has no second floor pair. So on it goes: 1,3,5,7,... – odd; 2,4,6,8,... – even.

Problem 7 *Is 9 odd or even? Why?*

Problem 8 *Using the above "construction" approach, find all the partitions of the number 5.*

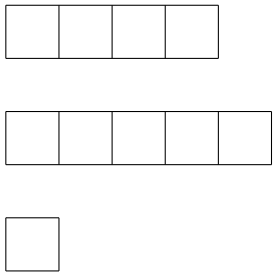
Problem 9 *Find all the partitions of the number 6 into odd parts. In other words, draw all the possible houses with 6 apartments and with an odd number of apartments on every floor. Hint: first, list all the positive odd integers less than 6.*

Problem 10 *Is 12 odd or even?*

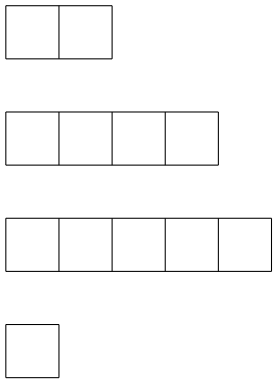
Example 2 *Compute*

$$12\text{-}4\text{-}5\text{-}1=$$

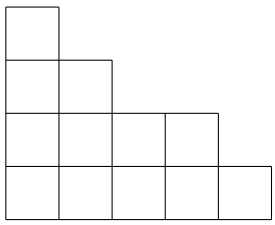
Let us first draw the numbers we need to subtract as floors "floating in the air".



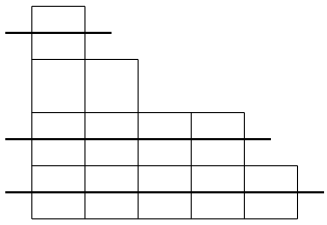
Let us add one more floor, also "floating in the air", so that the total number of the apartments becomes equal to 12.



We can solve the subtraction problem right now, but since our goal is not only to teach/learn subtraction, but also to familiarize ourselves with the above pictures, called Young diagrams, let's do two more steps. Let us assemble the "floating" floors into a proper house.



Finally, let us mark out the floors corresponding to the numbers we subtract. The last thing to do is to count the apartments of the remaining floor.



$$12\text{-}4\text{-}5\text{-}1=2$$

Problem 11

$$17\text{-}3\text{-}3\text{-}7\text{-}1=$$