FRACTIONS, DECIMALS, PERIODS, (IR)RATIONALS

MATH CIRCLE (BEGINNERS) 10/09/2011

(1) Here is a slightly different way to do long division and convert a fraction to a decimal. Let's try it with 3/7. First, multiply the numerator(=3) by 10, which gives you 30, then divide 30 by the denominator(=7), so you get 4, remainder 2. You can express this all on one line like so:

$$30 = 4 * 7 + 2$$

Next, take the remainder(=2), multiply *it* by 10, and divide the result again by 7. So the next line looks like this:

$$20 = 2 * 7 + 6$$

Continue on this way, multiplying each remainder by 10 and then dividing by 7 and writing it on one line. I've rewritten the first two lines below along with the third line. You do the next few lines. When should you stop? (Hint: Don't do more than 10 lines total... but you might not need to do that many...) When you're done, you will have computed the decimal expansion of 3/7 by looking at the quotients–the numbers just to the right of the equal sign. So the decimal expansion of 3/7 begins with 0.428).

$$30 = \mathbf{4} * 7 + 2$$

$$20 = \mathbf{2} * 7 + 6$$

$$60 = \mathbf{8} * 7 + 4$$

(2) Do the process above to find the decimal expansion of 13/15.

(3) Do the process above to find the decimal expansion of 3/22.

(4) An infinite decimal which eventually repeats itself is called **periodic**, and the digits in the repeating section is its **period**. For example:
   - the period of 1/3 = .33333333... is "3" and length of period is 1
   - the period of 41/333 = .123123123... is "123" with length 3
   - the period of 1/7 = .142857142857... is "142857" with length 6
   - the period of 5/6 = .833333333... is "3" with length 1

   (a) Jeff says, "When you use the technique above for converting a fraction to a repeating decimal, there are only 10 possible digits. Once you've used them all up, you have to repeat one of them, so the repeating section starts over. This means the length of the period of any repeating decimal you get from a fraction is at most 10." Is he right? Why or why not?

(b) What (if anything) forces the process used in problems 1-3 to eventually repeat itself?

(c) Without calculating it directly, how long could the period of a number of the form $\frac{k}{17}$ be? (k is any whole number.)

(d) What is the maximum possible length of period of a number of the form $\frac{1}{n}$, in terms of n?

(5) Check your answers to the previous problem by doing the process above to find the decimal expansion of 1/17. What is its period, and length of period?

(6) The decimal number

$$0.12345678910111213141516171819202122 23\ldots$$

is formed by writing down the digits of each whole number in order after the decimal point.

- Can you convert it into a fraction (ratio of whole numbers)? Why or why not?

(7) What is the 100th digit after the decimal point in the decimal representation of 1/7?

(8) What is the 2011th digit after the decimal point in the decimal representation of 1/7?

(9) What is the 1000th digit after the decimal point in the decimal from problem 6? ($0.1234567891011121314151617\ldots$)

(10) A *rational number* is a number which can be written in the form $\frac{a}{b}$, where a and b are both integers, and $b \neq 0$. (*Integers* are whole numbers including negatives and zero; in other words, the integers are the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.)

(a) Fill in the blank: The decimal representation of a rational number is always ____________________.

(b) Is the reverse true? That is, if a number has a decimal representation which is ____________________, does it mean that the number must be rational?

(c) Is 53 rational? Why or why not?

(d) Is -35.33 rational? Why or why not?

(e) Is zero rational? Why or why not?

(f) Can you write down or describe a number that's *irrational*–one which can't be written as a ratio of two integers? (Hint: There's been one already in this handout that you can use...)

(11) Given that $1/7 = 0.\overline{142857}$, what is the fraction represented by $0.\overline{285714}$?

(12) Given that $1/13 = 0.\overline{076923}$, what is the fraction represented by $0.\overline{923076}$?