CONTINUED FRACTIONS, CTD.

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Review. Recall that a finite continued fraction is an expansion of the form

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}},$$

where all the a_j's are integers, $a_0 \geq 0$, and $a_j \geq 1$ for $j \geq 1$. We also consider expansions that may not terminate:

$$[a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

Notation. The numbers a_j are called the *partial quotients* of the expansion. The fraction $\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$ is called the *kth convergent*.

Continued Fraction Expansions of Rationals. In all cases we have observed so far, the continued fraction expansion of a rational number has terminated.

(1) Compute the continued fraction expansion for $\frac{195}{154}$ using the worksheet provided. Do you see a pattern in the remainders r_j?
(2) In general, what is the relationship between a fraction $\frac{a}{b}$ and the first remainder? The first remainder and the second?
(3) Prove that the continued fraction expansion of any rational number terminates.

Quadratic Irrationals. A *quadratic irrational* is an irrational number which is the root of a quadratic equation $ax^2 + bx + c = 0$.

(1) The "golden ratio", $\varphi = \frac{1+\sqrt{5}}{2}$, satisfies

$$\varphi^2 = \varphi + 1, \quad \text{hence} \quad \varphi = 1 + \frac{1}{\varphi},$$

What happens when you repeatedly plug the right hand equation into itself?
(2) The number $1 + \sqrt{2}$ is sometimes jokingly called the "silver ratio". Can you guess why?

(3) We say that a continued fraction expansion is *eventually periodic* if the partial quotients a_j eventually fall into a repeating cycle. We use the notation $[a_0, a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_n}]$, e.g. $[1, \overline{2, 3}] = [1, 2, 3, 2, 3, 2, 3, 2, 3, \ldots]$.

See if you can find a number with each continued fraction expansion.

(a) $[3, \overline{3, 6}]$ (b) $[3, \overline{1, 2}]$ (c) $[0, 3, \overline{2}]$ (d) $[0, 1, \overline{10, 5}]$

(4) Prove that for any eventually periodic continued fraction expansion, there is a quadratic irrational with that expansion.

(5) Many people know the decimal expansion of π to several places. However, the continued fraction expansion is much less familiar!

(a) Use a calculator to find the first few (at least three) terms of the continued fraction expansion for π. What happens if you truncate the expansion after two terms? three terms?

(b) Compare: If you know the first three terms of the continued fraction expansion, how many decimal places of accuracy does this give?

(6) Given two fractions $\frac{a}{b}$, $\frac{c}{d}$, the fraction $\frac{a+c}{b+d}$ is called their *mediant*. What happens if you begin with the fraction $\frac{p_{k-2}}{q_{k-2}}$, and repeatedly take mediants with $\frac{p_{k-1}}{q_{k-1}}$? Try this out with some of the expansions you've computed.

(7) Prove that the convergents satisfy the recurrence relations

$$p_k = a_k p_{k-1} + p_{k-2}$$
$$q_k = a_k q_{k-1} + q_{k-2}$$

(Hint: Assume the relation holds for *any* continued fraction up to the kth level.)

(8) Use the previous exercise to prove the identity

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^n.$$

What does this tell you about $\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}$?

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
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6			
7			
8			

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0			
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0			
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0			
1			
2			
3			
4			
5			
6			
7			
8			

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec