Homework 5: Invariants and Geometric Constructions

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February 10, 2019

1 Homework

Problem 1.
Numbers $1, 2, \dots, 20$ are written at the board. Every operation erases two numbers a, b are replaces them by $a + b - 1$. What are all the numbers that could be left after the operation is applied 19 times?

Problem 2.
Given a $\triangle ABC$, let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB respectively. Show that the center of the circumcircle of $\triangle ABC$ is the same as the intersection of the altitudes of $\triangle A_1B_1C_1$. You may use the result of problem L5.5.

2 Reading

Solution 1 (H4.1)**.**
Let us number the positions of all coins from 1 to 100. In order to reverse the order of all the coins, the coin at position 1 has to end up at position 100. Now consider every single switching operation, and suppose we are switching coins at positions n and $n + 2$. Then coin at position n moves to position $n + 2$, coin at position $n + 2$ moves to position n, and the rest of the coins do not move. Then note that the parity of the position of each coin does not change. But then coins 1 could never move from position 1 to position 100, contradiction.

Solution 2 (L4.3)**.**
Solution 1:

Construct the midpoint M of AB, and draw a circle ω_1 with center M and radius MA. Then this is a circle with diameter AB. Now draw a circle with center A and radius CD, and consider an intersection point P of this circle with ω_1. Since P lies on the circle with diameter AB, we know that $\angle APB = 90°$. We also have $PA = BC$, which means that $\triangle PAB$ is our desired triangle.
Solution 2:

Draw an arbitrary line ℓ through an arbitrary point E, and construct a line m through E perpendicular to ℓ. Draw a circle with center E and radius BC, and let D be the intersection of m and this circle. Now draw a circle through D with radius AB, and let T be the intersection of this circle with ℓ. Then $ED = BC$, $DT = AB$ and $\angle DET = 90°$, which means the triangle $\triangle EDT$ is the right triangle we wanted to construct.