Lesson 2: Vieta’s formula

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Definition 1.
We say that \( x_0 \) is a root of a function \( f(x) \) if \( f(x_0) = 0 \).

Problem 1.
a) Let \( ax^2 + bx + c = 0 \) be a quadratic equation. Show that if it has two distinct real roots \( x_0, x_1 \), then \( ax^2 + bx + c = a(x-x_0)(x-x_1) \). Hint: consider the difference between \( ax^2 + bx + c \) and \( a(x-x_0)(x-x_1) \). What degree is it? How many roots does it have?

b) Show that a quadratic equation cannot have more than two distinct real roots.

c) Now suppose that \( ax^2 + bx + c \) has exactly one real root \( x_0 \). Show that \( ax^2 + bx + c = a(x-x_0)^2 \).

Problem 2.
a) [Vieta’s formulas] Consider a quadratic equation \( ax^2 + bx + c = 0 \) with real roots \( x_0, x_1 \). Show that

\[
x_0 + x_1 = -\frac{b}{a}
\]

\[
x_0x_1 = \frac{c}{a}
\]

These are called Vieta’s formulas. You may use the result of problem 1 even if you did not solve it.

b) Given any two real numbers \( x_0, x_1 \) with \( x_0 + x_1 = u \) and \( x_0x_1 = v \), show that both \( x_0 \) and \( x_1 \) are roots of the quadratic equation \( x^2 - ux + v = 0 \).

Problem 3.
a) Let \( x_0, x_1 \) be roots of a quadratic equation \( ax^2 + bx + c = 0 \). Find the formula for \( x_0^2 + x_1^2 \) in terms of \( a, b, c \).

b) Let \( x_0, x_1 \) be roots of \( x^2 + bx + c = 0 \). Find the formula for \( x_0^3 + x_1^3 \) in terms of \( b, c \).
Problem 4.
Consider a circle whose diameter is the side $AB$ of the triangle $ABC$. Show that if that circle contains the midpoint of $AC$, then $\triangle ABC$ is isosceles.

Problem 5.
Let $AC$ be a diameter of a circle, and $B$ be a point on the circle distinct from $A$ and $C$. Let $P$ be the foot of the perpendicular from $A$ to the tangent to the circle at $B$. Show that $AB$ is the angle bisector of $\angle PAC$. 