Algebraic Numbers

Learn how to transcend the Real

Math Circle

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Today we are going to talk about numbers. Exciting right? Well even if you don't think so right at this second, you might be a bit more interested by the end of the lesson. Before we move on, let's review a few facts.

1. Let's review some facts about rational numbers first.

   (a) First, a very easy question. What is the definition of a rational number in terms of it's fractional representation? *Hint, this is not a trick question, and should be very easy.

   (b) How could you define a rational number in terms of its decimal expansion? *Hint, this question is kind of easy, but make sure that you give a precise answer.

(c) Are the following numbers rational or irrational? Answer using your intuition, and make sure that this jives with your definition as well. If your definition doesn't agree with your intuition, change one of the two until they agree! The underbar notation means repeated forever, so $1.23\underline{456} = 1.23456456456456\ldots$ Ask an instructor if this is unclear.

Number	Intuitively Rational	Rational Using My Definition
$1.0000\underline{0}$		
$1.0101\underline{01}$		
$0.123456789101112\ldots$		
$0.\underline{123456789}$		
$0.1284718\underline{123}$		
$0.11010010001\ldots$		
$3.1415926\ldots$		

(d) There are a LOT of misconceptions when it comes to rational and irrational numbers. Please explain why the following statements are wrong.

"An irrational number is a number without a pattern."

"A number is rational if its decimal part repeats over and over."

"If a number is irrational, then you can find any finite sequence of digits in its decimal expansion."

I can't tell you how many times I've heard the following "Because π is irrational, that means that if you converted Romeo and Juliet to a sequence of numbers, then you could find it somewhere in the expansion of π."

Most real numbers are rational.

(e) **It is (relatively) easy to convert from the fractional representation of a number to it's decimal expansion. Doing the reverse is less straightforward, but is possible. Let $r \in [0, 1]$ be a number whose decimal expansion is of the form $r = 0.p_1p_2p_3 \ldots p_n\underline{d_1d_2 \ldots d_m}$ where p_i, d_j are one of the 10 digits. Show that $s = p_1 \ldots p_n - r \cdot 10^n = 0.\underline{d_1 \ldots d_m}$. Further, show that $d_1 \ldots d_m - s \cdot 10^m = s$. Finally use these two facts to develop an algorithm for converting decimal representations to fractional representations.**

2. Ok, now we are going to watch today's video! This video is from the channel Vihart, and is called 'Transcendental Darts'.

`https://www.youtube.com/watch?v=Swm8tTLWirU`

This video is very informative, but goes *very* quickly. Let's look at a few points brought up in the video more closely.

(a) The first thing discussed in the video is the following question. If I pick a number uniformly randomly within the interval $[0, 1)$, what are the chances that it is a rational number? The answer is zero, but this is not obvious. Can you show that the chance of picking the number $1/2$ is exactly 0? *Hint, a very useful way to show that a non-negative number is zero, is to show that it is smaller then every positive number. Use this principle, combined with the fact that the chances of choosing the point $\frac{1}{2}$ is smaller than the chances of picking a number in the interval $[1/4, 3/4]$ and $[3/8, 5/8]$ and $[7/16, 9/16]$ etc...

(b) **Can you use the above argument to show that the chances of picking any rational number is zero?**

(c) Now let's get to what I think is the spirit of the video. An algebraic number is a number which can be written as the root of a polynomial, where the polynomial has integer coefficients (the zero polynomial doesn't count). Using only the definition, can you prove that $\sqrt{2}$ is algebraic? What about $\sqrt[3]{2}$?

(d) For the following algebraic numbers, find a polynomial (with integer coefficients) which they are roots of. I even did the first one for you! What a nice guy.

i. $1-\sqrt{2}$. If $x = 1-\sqrt{2}$, then $x-1 = -\sqrt{2}$, and $2^2 - 2x + 1 = 0$ and so finally $x^2 - 2x - 1 = 0$

ii. $\sqrt{1-\sqrt{2}}$.

iii. $3^{7/12}$.

iv. $\frac{1}{4}$

v. $\sqrt[7]{\sqrt{3}-\sqrt{2}}$

3. Now let's prove some properties about algebraic numbers.

   (a) One way to express a number is by expressing it's continued fraction. When you write a number as it's continued fraction, you write it in the form $x = [a_0, a_1, \dots]$ where all of the a_i are integers, and

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

   Kind of like how writing a number in decimal form $x = b_0.b_1b_2\dots$ is the same as

$$x = b_0 + b_1 10^{-1} + b_2 10^{-2} + \dots.$$

   Can you find the continued fractions for the numbers $8/3, -27/2$ and $110/111$?

(b) Prove that any rational number's continued fraction only has a finite number of non-zeroes. In other words, after a certain point the continued fraction is just a string of 0s forever.

(c) Compute the continued fraction of the number $\phi = \frac{1+\sqrt{5}}{2}$ and $\sqrt{2}$.

(d) Can you prove that if a number's continued fraction repeats forever then it is an algebraic number? Hint* This is kind of like problem 1.e.

(e) Can you prove that all numbers are of this form?

(f) **You are given a very special (and unusual) hole puncher. This puncher punches out points on the x-y plane. If you hover this puncher over a point (x, y), then it will punch out all points that are an algebraic distance away from the point (x, y). How many punches do you have to make to punch out all points in the x-y plane?**