Combinatorics

Count your blessings

without replacement

Math Circle

April 28, 2018

Combinatorics is one of the major fields of mathematics, and concerns itself with counting! However, combinatorics proofs are of a certain flavor. Combinatorial proofs *almost always* make use of bijections.

1. Each set of problems should follow a common theme. Note the empty space between questions, to give the students space to work out the answer. Let's remember what a bijection is.

   (a) What is a bijection? Hint, it's a function between two sets (the sets usually represent two different things), but what is special about these functions?

   (b) You (better!) remember that we spent a lot of time talking about bijections when we were talking about infinities. What purpose did bijections serve there? Why did we care about them at all?

(c) If I have two sets $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ and know that there exists a bijection from $\mathcal{A}$ to $\mathcal{B}$, then what can I say about n and m?

(d) **A function f between two sets $\mathcal{A}, \mathcal{B}$ is called surjective, if for every $y \in \mathcal{B}$, there is at least one $x \in \mathcal{A}$ such that $f(x) = y$ (Note, x does not have to be unique). Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite. Prove that if there are two surjective functions which go from $\mathcal{A}$ to $\mathcal{B}$ and vice versa, then there is a bijection from $\mathcal{A}$ to $\mathcal{B}$.**

(e) **Can you modify your proof to work in the case where $\mathcal{A}$ and $\mathcal{B}$ are not necessarily finite?**

2. Saying that combinatorics is about counting is technically true, but kind if misses the point. A good combinatorial proof shows that two sets which are constructed in different ways have the same size by finding a bijection between them. This means that many times combinatorists do not use proof techniques such as contradiction or induction, and eschew algebra where ever possible. Instead, they try to rely on bijections as as much as possible. The belief is that bijections provide a lot of insight, which is the real goal of the proof in the first place.

   With that in mind, try and find a combinatorial proof of the following:

   (a) Remember what the expression $\binom{n}{k}$ means? Write it down.

   (b) If t_n is the n'th triangular number, show that $t_n = \binom{n+1}{2}$. You can probably do this inductively, or algebraically, but I want you to show it combinatorially, i.e. by forming a bijection.

   (c) Show that $\binom{n}{k} = \binom{n}{n-k}$.

(d) Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ if $n > k > 0$.

3. Now, let's take a look at a very famous math problem that can be solved combinatorially. Today we are going to be watching a video from Numberphile again, this one is called 'Stars and Bars (and bagels) - Numberphile'.

   `https://www.youtube.com/watch?v=UTCScjoPymA`

   (a) Were you paying attention? Where in the video did Prof. Ribet use a bijection? What was his bijection?

   (b) Find the number of ways to choose 12 bagels out of 5 varieties if you must order at least one of each variety. What if you have to order at least 2 of each kind?

(c) A *monotonic path* from $(0,0)$ to (n,m) where $n,m \in \mathbb{N}$ is a path starting from $(0,0)$ and ending at (n,m) which consists solely of moves up, or to the right by 1. Compute (combinatorially) the number of monotonic paths from $(0,0)$ to (n,m).

(d) Suppose that we have a monotonic path from $(0,0)$ to (n,n). Define the exceedance of a path as the number of times that we move up, after we are already above the line $y = x$. Let P be a path with exceedance greater than one. Find a bijection between the paths with exceedance e and exceedance $e-1$?

(e) How many paths are there with exceedance 0? This number of called the n'th Catalan number.

(f) The parenthesis (()) and (()()) and ()()(((())())) are all valid, but)(and ())(are not. A sequence S of (and)'s is called valid under two conditions. First, that there are the same number of ('s and)'s. What is the second condition?

(g) Before starting this question, check to make sure with an instructor to make sure that you got the last one right. How many valid strings of $2n$ parenthesis are there? Prove it combinatorially.

4. Now here are some bonus problems for the strong of heart. Many of these are taken from the excellent (and huge) book Enumerative Combinatorics by Prof Richard P Stanley.

   (a) Find combinatorial proofs of the following: $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

(b) $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ when $n \geq 1$.

(c) **Challenge.** $\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$.

(d) **Challenge.** $\sum_{i+j+k=n}\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k} = \sum_{r=0}^{n}\binom{2r}{r}$.