Game theory

Less like a game of Chess; more like a game of Risk

Math Circle

April 22nd, 2018

An extremely interesting field of math (in my humble opinion anyways) is the field of game theory. The field of game theory studies the question of why people do what they do.

1. Let's warm up today with some brain teasers. To make the discussion as straightforward as possible, we will make the (dubious) assumption that all people in our games are perfect mathematicians, and will always make the decision that is best for them.

   (a) Four pirates (Alice is the Captain, Bob the first mate, Charlie the navigator and Diana a deck hand) are sailing the high seas and dig up 100 gold coins. They decide to share the loot between themselves. The protocol for sharing loot is as follows. Alice proposes a division of the coins (e.g. 45 for me, 25 for Bob, 15 for Charlie and Diana). All of the pirates vote on whether they like the division or not. If the vote fails, then the one who proposed the division walks the plank, and the game proceeds with the next in command proposing a new division. If the vote ties then it's a failure. Without doing any math, who do you think is best off in this system?

   (b) If Diana is the last one standing, then she gets all of the loot. If it is down to just Charlie and Diana, what division can Charlie propose that Diana would accept? Why?

(c) Now suppose that Alice has already walked the plank. Prove that Diana should *never* accept Bob's division. What about Charlie? What divisions should he accept/reject?

(d) How will the pirates split the loot? Who will live and die?

(e) How does the above analysis change a tie vote passes?

2. Before we get into the famous prisoner's dilemma, let's do some more warm up problems.[1]

   (a) Pair off. One of you or your partner are player 1, the other is player 2. You two are going to play rock-paper-scissors, under the special rule that player 1 can't throw scissors, and player 2 can't throw rock. Which player has the advantage? Play 10 games with your neighbor and record the results of your games.

   (b) If you win, you get one point. If you lose, then you get -1 point. If you tie you get zero. Write down the payoff matrix for this game below.

   (c) Suppose that both players choose between their options randomly. What is the expected number of game that player 1 would win after 10 games? Player 2?

[1] These questions are taken from a previous handout by Mr. Brent Woodhouse

(d) Suppose that player 1 chooses rock with probability p, and player 2 chooses paper with probability q. If player 1 doesn't know q, what value of p guarantees that they get the maximum payoff? Same question for player 2.

3. Time to watch today's video! This video is from a channel called SciShow, and is called "Game Theory: The Science of Decision-Making." We are going to focus mostly on the prisoner's dilemma today. Maybe in a future class we'll talk about cooperative games.

   `https://www.youtube.com/watch?v=MHS-htjGgSY`

   (a) Let's say that the payoff for the prisoner's dilemma is given below. Player 1 first coordinate, and player 2 the second.

	Keep Quiet	Defect
Keep quiet	(-1,-1)	(-5,0)
Defect	(0,-5)	(-2,-2)

   Explain in words why you can be so sure that both people will defect.

(b) Let's say that there was a new DA in town who wanted to be tougher on crime. He changed the rules, so that even if you defected you still got a payoff of -1. Explain how this would (paradoxically) cause more people to spend less time in jail.

(c) Let's say that before the thieves ever commit their crime they have the option of joining the mob. **The mob will not help them in any way.** If they join the mob, then anyone who defects in the future will have their house burnt down (a payoff of -10 for that person). Why would anyone join the mob? Aren't they better off not joining and having a zero percent risk of a burnt house?

(d) These past two examples should have seams a little paradoxical. Let's think of a real world example. There are two companies who fish a lake. The lake is big enough that if only one of the companies over fished, then there would be enough fish for everyone. If both companies over-fish, the fish population would plummet. Explain why it would be in both companies best interest to pass a law which makes over fishing illegal, even though it could only hurt them.

4. Now, let's talk about repeated games. This was the subject of the little game that I had you play.

   (a) Tariffs are special taxes that are levied on imported goods from specific countries. China and the US have lots of tariffs between each other, for example the US has a tariff on Chinese steel and aluminum, and China has a tariff on lots of US food and fruit. This makes it more difficult for China to export steel, and the US to export fruits. If tariffs benefit the country imposing the country a little, and hurt the target country a lot, explain how the problem of levying tariffs can be viewed as a kind of prisoner's dilemma. Come up with your best guess of a payoff matrix.

   (b) From a game theory perspective, what is the 'correct' move for a country to make when it comes to tariffs?

   (c) Suppose that the US and China each have 5 tariffs between each other that they would like to lift. They have to take turns removing tariffs one at a time simultaneously. Further, suppose that the politicians in both countries are honest, but will not lift a tariff if they think that the other country will not lift their tariff (that is, they will only cheat if they think that the other country will cheat). Also, if one country cheats the other, then that is the end of negotiations. What is the 'correct' move for both countries to make? Prove it.

(d) **Same situations as before, but this time after each round of negotiations there is a p chance that negotiations end no matter what. For what value of p will the countries cooperate? Use your payoff matrix from question 4.a**

5. Finally, here are some more questions about repeated games, which are more obviously mathematical.

   (a) **Suppose that two people agree to meet in a Starbucks in Westwood, but forget to specify which one. Suppose that there are 2 Starbucks total, and by happenstance they meet at the wrong ones to start. Both people's phones are dead. Each person chooses a fixed probability $0 < p < 1$. Each persons choice of p is the same. With probability p every 10 minutes each person makes their way over to the other Starbucks, and with probability $1 - p$ they stay put. In terms of p, how many minutes will it take on average for the friends to reunite?**

(b) **What value of p should the friends choose so that the expected waiting time is minimal?**

(c) **Same question, but what if there are three Starbucks?**