Lesson 2 Problem 1 Solution

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Problem 1.
Suppose the sum after n operations is S_n. We want to figure out the sum after $n+1$ operations. During an operation, we take every pair of numbers and insert their sum between them. Let the sequence after n operations be $x_1, \ldots, x_m$. Then every number x_k in the sequence contributes to the sum $x_{k-1}+x_k$ which will go between x_{k-1} and x_k, and to the sum $x_{k+1}+x_k$. Well, every number except for x_1 and x_m, which only contribute to one sum each since they don't have a left or right neighbor respectively. Then, if we count the sum of numbers after the $(n+1)$'st operation, every number except for x_1 and x_m will be counted 3 times – once by itself and twice in the sums with neighbors. x_1 and x_m will be counted twice each. So the sum after $(n+1)$'st operation will be $3S_n - x_1 - x_m$. But the first and last number are always 1, so we get $S_{n+1} = 3S_n - 2$.
Now we wish to show by induction that $S_n = 3^n + 1$. The base case S_1 checks out: $1 + 2 + 1 = 3^1 + 1 = 4$. For the inductive step, we want to show that $S^{n+1} = 3^{n+1} + 1$ if we know that $S_n = 3^n + 1$. Substituting $3S_n - 2$ for S_{n+1}, we get

$$S_{n+1} = 3S_n - 2 = 3(3^n + 1) - 2 = 3^{n+1} + 1$$

exactly as desired. Induction is complete, and now plugging in $n = 100$ we have solved the problem.