Counting Vertices, Edges, Faces (Euler's Formula of Polyhedra)

March 4, 2018

1 Warm-up:

In mathematics we commonly need to prove things. We have many different techniques to prove things, but sometimes it is more important to disprove something. In order to disprove something, a counterexample is very helpful.

Try and find a counterexample to the following statement: "If a number p is prime, then the number p+2 is also prime".

Your counterexample:

Some of the shortest papers in mathematics have been simply counterexamples (see "Counterexample to Euler's Conjecture on Sums of Like Powers" by Lander and Parkin 1966)

2 Polygons on the plane

A *polygon* is a shape on the plane with the following properties:

-
-
-
-

Answer the following questions after experimenting with polygons with a small number of sides (triangles (3 sides), quadrilaterals (4 sides), pentagons (5 sides), etc.):

1. Suppose we have a polygon; add one more vertex to it. Draw a picture of this.

2. How does the number of edges change in this situation?

3. Suppose we have a polygon; add one more edge to it. Draw a picture of this.

4. What is the relation between the number of vertices (V) of a polygon and the number of its edges (E)?

3 Vertices, Edges and Faces of Polyhedra

Complete the following definition of a polyhedron by finishing the sentences:

A *polyhedron* is an object in 3dim space with the following properties:

- polyhedron has vertices, edges and faces;
- *faces* are ;
- *edges* are ;
- *vertices* are ;
- polygons are joined along ;
- if two faces share a part of an edge, they share the entire edge;
- all together, a polyhedron surrounds a piece of 3-dim space;

1. Mark two faces on a cube that share an edge.
2. Mark two faces on a pyramid that share exactly 1 vertex.
3. Find an example of a polyhedron where there are 10 edges connected to a vertex. Make a picture.
4. How many faces meet at the vertex in the previous question?

- Collect data and fill in the following table:
  We saw the following last time:

#	Polyhedron		F(aces)	V(ertices)	E(dges)
1	cube				
2	triangular prism				
3	5-prism				
4	pyramid				
5	tetrahedron				
6	octahedron				
7	"tower"				
8	cube with a cut corner				
9	(your own)				

Explore the relations between the number of vertices, edges and faces by looking at the polyhedra in your table.

1. Is the following true or false:
   The bigger the number of faces, the bigger the number of vertices
   If it is false, find a pair of polyhedra that disproves this statement.

2. Is the following true or false:
   The bigger the number of edges E, the bigger the number of vertices V?
   If it is false, find a pair of polyhedra that disproves this statement.

3. Is the following true or false:
The bigger the number of edges E, the bigger the number of faces F?
If it is false, find a pair of polyhedra that disproves this statement.

4. Is the following true or false:
The bigger the number of edges E, the bigger the sum $V+F$?
If it is false, find a pair of polyhedra that disproves this statement:

5. Rearrange the polyhedra in the table so that the number of faces F increases:

#	Polyhedron (name)	F	V	E

Based on your findings in 4 and 5, can you make an observation about the relation between V, F and E that is true for all the polyhedra in the table?

4 Testing the conjecture

Based on the table, we conjecture that the following relation between the number of edges, faces, and vertices is true for ALL polyhedra:

We will look at two different ways of modifying a polyhedron*: building a roof* and *cutting a corner.*

4.1 Building a roof

Take a polyhedron. We will *build a roof* over one of its faces.

- pick a face of the polyhedron and add one vertex outside of the original polyhedron, so that it is "over" the chosen face;
- connect all the vertices of the chosen face to the new vertex.

A roof over a triangular face:

1. Make a picture of a roof over a triangular face. First draw a triangle (the face). Then add a point abovr the trangle. Connect all the vertices of the triangle with the new vertex.

2. Count how the numbers of vertices, edges and faces changed:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (roof added)			

3. Is the conjecture true for the new polyhedron?

A roof over a quadrilateral face:

1. Make a picture of a roof over a square (or any other quadrilateral) face. First draw a square (the face). Then add a point above the square. Connect all the vertices of the square with the new vertex.

2. Fill in the table below to see how the numbers of vertices, edges and faces changed after we built the roof:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (roof added)			

3. Is the conjecture true for the new polyhedron?

4.2 Cutting a corner

Suppose we have a polyhedron for which our conjecture is true. Let's *cut a corner* next to one of its vertices:

- pick a vertex of the polyhedron and make a flat cut across all the edges connected to this vertex;
- remove the part that was cut off to get the new polyhedron.

Cutting a corner across 3 edges

1. Draw a picture of the process of cutting a corner. Draw a vertex and 3 edges coming out of it. Pick a point on each of these three edges. Cut across so that these 3 points become new vertices:

2. To see if the conjecture is true for the new polyhedron, let's see what happens with the number of vertices, edges and faces when we cut a corner:

Polyhedron	Faces	Vertices	Edges
Old	F	V	E
New (=Old without a corner)			

3. Is the conjecture true for the new polyhedron?

4. Suppose you cut a corner from a polyhedron. Explain how you can reconstruct the original polyhedron:

4.3 Testing our Conjecture on a Different Type of Polyhedra

To see if our conjecture is true in more cases, we will pick a very different polyhedron.

Let's consider a very different polyhedron. For example, take a rectangular "picture frame".

1. Build a model of the rectangular "picture frame" out of connectable cubes.

2. Count the number of vertices, faces and edges.

$$F =$$
$$V =$$
$$E =$$

3. Is your conjecture still true for this "picture frame"?

4. Can you explain how the "picture frame" is different from all the previous polyhedra we have considered?

5. Make a conclusion about the relation of the number of faces (F), vertices (V) and edge (E). Explain for what type of polyhedra it is true. Do you think

Congratulations! You have rediscovered Euler's famous formula:

$$F + V = E + 2.$$