MATH CIRCLE HIGH SCHOOL 2 GROUP, WINTER 2018
WEEK 7: NUMBER THEORY, PART 3

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1. Series

Last time, we used Big Oh notation; let's refresh:

Definition. Big Oh Notation *is a way of saying that one sequence's growth is bounded by another; if if there are some constants N, c such that for all $n > N$, $f(n) < c \cdot g(n)$, then we write $f(n) = O(g(n))$. This then lets us write $f(n) = h(n) + O(g(n))$ to mean $f(n) - h(n) \ll g(n)$, which can be interpreted as "$f(n)$ is approximated by $h(n)$ with an error of order at most $g(n)$." Such equations can be manipulated; multiplying both sides by a term, or adding a term to both sides, maintains the equality.*

If we have both $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then we say that $f(n) = \Theta(g(n))$; this means that there is some constant α such that $f(n)/g(n)$ goes to α as n goes to infinity.

Problem 1. *Prove that $H(n) = \sum_{i=1}^{n} \frac{1}{n} = \Theta(log(n))$; if you have taken calculus, you might have an easy way to do this (in particular, by first noting that $H(n) = \Theta(H(n+1))$, and then giving upper and lower bounds for $log(n)$). If not, try to find upper and lower bounds for $H(2^k)$ in terms of $k = log_2(2^k)$, and noting that the natural log is a multiple of the base-2 log.*

The sum of the reciprocals of the integers diverges; the sum of the reciprocals of the squares converges. This gives a first-order way to estimate how frequently a sequence appears in the integers; in particular, the primes are more common than the squares:

Problem 2. *Prove that the series $\sum_{i=1}^{\infty} \frac{1}{p_i}$ summing up the reciprocals of all the prime numbers diverges as follows:*

- *Assuming that $\sum_{i=1}^{\infty} \frac{1}{p_i}$ converges, explain why there must be some k such that $\sum_{i=k+1}^{\infty} \frac{1}{p_i} < \frac{1}{2}$.*

- *For any positive integer x, let M_x denote the set of integers in $\{1, 2, \ldots, x\}$ which are not divisible by any prime greater than p_k. Show that $|M_x| \leq 2^k \sqrt{x}$ (Hint: a number in M_x is a square number, times some number of primes, each of which is at most p_k. How many ways are there to choose each of those?)*

- *Show that $\frac{x}{2} < |M_x|$ by over-counting the number of integers at most x which* aren't *in M_x. (Hint: in particular, each must be divisible by at least one prime greater than p_k; how many numbers less than x are there for each such prime?)*

- *Conclude that, for sufficiently large x, we have a contradiction.*

So, now that the primes are reasonably dense, the question becomes: exactly how dense?

Definition. *The* prime counting function $\pi(x)$ *is defined to be the number of distinct prime numbers less than or equal to* x*.*

One of the great results of number theory is the Prime Number Theorem, which says that the ratio between $\pi(x)$ and $\frac{x}{log(x)}$ approaches 1 as x goes to infinity. This result is beyond our scope today, but we will prove that $\pi(x) = \Theta(\frac{x}{log(x)})$.

Problem 3. *Prove that* $2^{2n} > \binom{2n}{n}$*.*

Problem 4. *Prove that the product of all prime numbers between* n *and* $2n$*,* $\prod_{n<p\leq 2n} p$ *divides* $\binom{2n}{n}$*.*

Problem 5. *In terms of the prime counting function, what is a number* k *such that* $n^k < \prod_{n<p\leq 2n} p$*?*

Problem 6. *Put together the past three problems into an upper bound for* $\pi(2n) - \pi(n)$*.*

Problem 7. *Prove, by induction on n, that $\pi(n) = O(\frac{n}{log(n)})$. Note that the coefficients cannot increase as n increases! It is feasible to show that $\pi(n) \leq 2\frac{n}{log(n)}$, but feel free to replace 2 by a larger number if that's easier.*

To prove the reverse bound, we'll need some new notation.

Definition. *If a is a real number, the* integer part of a, *$[a]$, is the largest integer such that $[a] \leq a$. For any prime number p and integer n, $v_p(n)$ is defined to be the largest power of p dividing n.*

Problem 8. *Show that*

$$v_p(n!) = [n/p] + [n/p^2] + [n/p^3] \ldots = \sum_{j=1}^{\infty} [n/p^j]$$

(Hint: note that, for any given n, this is actually a finite sum, plus an infinite number of 0's.)

Problem 9. *Prove that for any real numbers a and b, $[a+b] - [a] - [b]$ is either 1 or 0.*

Problem 10. *Prove that for any integers n and k, $p^{v_p(\binom{n}{k})} \leq n$. (Hint: first express $v_p(\binom{n}{k})$ in terms of $v_p(n!)$, $v_p((n-k)!)$, and $v_p(k!)$, then use this to get an upper bound for $v_p(\binom{n}{k})$ in terms of $log(n)$ and $log(p)$)*

Problem 11. *Prove that* $\binom{n}{k} \leq n^{\pi(n)}$ *for any* n *and* k.

Problem 12. *Prove that* $2^n \leq (n+1)n^{\pi(n)}$ *for any* n.

Problem 13. *Prove that* $\frac{1}{2}\frac{n}{log(n)} < \pi(n)$ *for any* $n \geq 15$*; then conclude that* $\pi(n) = O(\frac{n}{log(n)})$.