MATH CIRCLE HIGH SCHOOL 2 GROUP, WINTER 2018
WEEK 2: MORE FINITE GEOMETRY

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1. Projective Planes

Definition. *A* Finite Projective Plane *is a geometry satisfying the following axioms:*

- *Every two distinct points determine a unique line.*
- *Every two distinct lines meet at a unique point.*
- *There exist four points, no three of which are on the same line.*
- *There are only finitely many points.*

Theorem 1. *In any finite projective plane, the following hold, for some positive integer n (called the* order *of the finite projective plane):*

(1) *All the points of the plane are on all the lines through any given point.*
(2) *Every line contains $n + 1$ points.*
(3) *The total number of points in the plane is $n^2 + n + 1$.*
(4) *Every point is on $n + 1$ lines.*
(5) *The total number of lines in the plane is $n^2 + n + 1$.*

Problem 1. *Draw an example of a Finite Projective Plane*

Problem 2. *A city bus network has the property that one can get from any bus stop to any other without transferring, for any pair of routes there is exactly one stop where one can transfer between them, and there are exactly three bus stops on each route.*

How many bus routes are there in town?

Problem 3. *Prove that in a projective plane, there are four lines, no three of which pass through the same point.*

Problem 4. *Prove that all the lines in a projective plane have the same number of points as follows:*

(1) *Pick a line m, and say that it has k points on it. Explain why there must be at least two points not on that line.*

(2) *Call the two points not on m P and Q; explain why there must be a line other than m that doesn't go through both P and a point on m (to do so, it will go through Q); call this line l.*

(3) *Explain why each of the lines that contain P and intersect m must intersect l at a different point. How many points does this imply that l has?*

(4) *Reverse the above between* m *and* l *to get the other inequality.*

(5) *Explain why any line not passing through* P *must have exactly* k *points.*

(6) *Explain why any line not passing through* Q *must have exactly* k *points.*

(7) *Explain why* PQ *must have exactly* k *points.*

Problem 5. *In a particular city, there are 57 bus routes. One can get from any bus stop to any other, for any pair of routes there is exactly one stop where one can transfer between them, and every route has at least 3 stops. How many stops are there on each bus route?*

Problem 6. *Prove that for any theorem about a projective geometry, if you switch the words "line" and "point," you get another true statement. (Hint: prove that the switched version of each axiom holds. Then reason that since all theorems come from the axioms, however you would prove one theorem, you could use the switched axioms to prove the other.) Use this to prove the parts of Theorem 1 that you haven't already shown.*

Problem 7. *In the game* Spot It!*, two cards with some number of symbols on them are placed out, and players must determine what symbol is on both cards. So, for the game to work, each pair of cards must share exactly one symbol. So as to not be wasting space, each symbol needs to be on at least two cards. And, to prevent a too-easy round, each card needs to contain at least two symbols. And, to prevent the empty game, there must be some number of symbols. Lastly, each card must have the same number of symbols.*

(1) *Draw a collection of cards that follows all the rules of* Spot It! *except for the last.*

(2) *Give an example of a potential collection of* Spot it! *cards that follows all the rules of* Spot It!. *but if you replace cards and symbols with lines and points, you don't get a projective plane.*

(3) *Give one additional condition on* Spot It! *cards that makes the rules of* Spot It! *equivalent to the axioms of a finite projective plane. Prove that equivalence.*

(4) *How many cards are there in a* Spot it! *deck?*

2. Affine Spaces

Definition. *A* Finite Affine Plane *is a geometry satisfying the following axioms:*

- *Every two distinct points determine a unique line.*
- *Any point not on a line l is on precisely one line that does not intersect l.*
- *There exist four points, no three of which are on the same line.*
- *There are only finitely many points.*

Theorem 2. *In any finite affine plane, the following hold, for some positive integer n (called the* order *of the finite affine plane):*

(1) *If two lines are parallel to a third, they are parallel to each other.*
(2) *Every line contains n points.*
(3) *The total number of points in the plane is n^2.*
(4) *Each line intersects n^2 other lines.*
(5) *Each line is parallel to n lines (including itself).*
(6) *The total number of lines in the plane is $n^2 + n$.*

Problem 8. *Draw an example of a Finite Affine Plane.*

Problem 9. *Prove that all the lines in a Finite Affine Plane must have the same number of points.*

Problem 10. *Prove that there cannot be seven distinct straight lines in a Euclidean plane so that there are at least six points where three of them intersect, and at least four points where exactly two of them intersect.*

Problem 11. *The game* Set *consists of a deck of cards, each with 1, 2, or 3 symbols. The symbols on a card are all red, green, or purple. They are all ovals, diamonds, or squiggles. They are all filled in, shaded, or empty. To score in the game, you must find 3 cards such that, in each aspect (number, color, shape, and shading),either all three are the same, or all three are different. 3 such cards are called a "set."*

While there are 81 cards in a set deck (one for each possible choice of characteristics), can you find a collection of cards such that, if the cards are points and the "sets" are lines, you end up with an affine plane?