MATH CIRCLE HIGH SCHOOL 2 GROUP, WINTER 2018
WEEK 5: NUMBER THEORY, PART 2

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1. Arithmetic Functions

Definition. *The* sigma function $\sigma(n)$ *is defined as the sum of the divisors of the positive integer* n.

$$\sigma(n) = \sum_{d|n} d$$

The divisor function $\tau(n)$ *is defined as the number of divisors* n *has.*

$$\tau(n) = \sum_{d|n} 1$$

The Euler totient function $\phi(n)$ *is defined as the number of relatively prime numbers less than or equal to* n.

The unity function $u(n)$ *is defined as* $u(n) = 1$.

The unit function $e(n)$ *is defined as* $e(1) = 1$, *and* $e(n) = 0$ *for* $n > 1$.

The identity function $N(n)$ *is defined as* $N(n) = n$.

The Mobius function $\mu(n)$ *is defined as* $\mu(n) = 0$ *if* n *is divisible by a square larger than* 1, *and* $\mu(n) = (-1)^k$ *if* n *is square-free, where* k *is the number of prime factors of* n.

Definition. *Given two arithmetic functions* f, g, *we define their* convolution *as*

$$f * g(n) = \sum_{d|n} f(d)g(n/d)$$

Problem 1. *What is* $u * u(n)$*?*

Problem 2. *What is* $N * u(n)$*?*

Problem 3. *What function $g(n)$ has the property that $f * g(n) = f(n)$ for all arithmetic functions f and positive integers n?*

Problem 4. *What function $g(n)$ has the property that $g * u(n) = e(n)$ for all positive integers n?*

Problem 5. *What function $g(n)$ has the property that $g * u(n) = N(n)$ for all positive integers n?*

Problem 6. *Confirm that convolution is both commutative and associative.*

Problem 7. Mobius Inversion *on arithmetic functions is the following statement: if f and g are arithmetic functions, then*

$$f(n) = \sum_{d|n} g(d)$$

is true if and only if

$$g(n) = \sum_{d|n} f(d)\mu(n/d)$$

(1) *Re-phrase Mobius inversion just in terms of convolution of functions, without any sums.*

(2) *Prove that mobius inversion holds for all functions.*

2. Bounds on Arithmetic Functions

Definition. *We say that $f(n) \ll g(n)$ if there are some constants N, c such that for all $n > N$, $f(n) < c \cdot g(n)$.*

Problem 8. *Show that $\sigma(n) \ll n * log(n)$ (you can use the fact that $\sum_{k=1}^{n} \frac{1}{k} \leq log(n)$.)*

Problem 9. *Show that $\tau(n) \ll \sqrt{n}$ (in fact, it's actually true that $\tau(n) \ll n^{1/k}$ for any $k > 0$, but that's harder to prove.)*

Problem 10. *Disprove that $\sigma(n) \ll n$ (This means that, for any constant coefficient c, you need to come up with arbitrarily large counterexamples.)*

Problem 11. *Disprove that $\tau(n) \ll 1$.*

3. Average Values of Arithmetic Functions

Definition. Big Oh Notation *is another way of expressing our previous comparison; if* $f(n) \ll g(n)$*, then we write* $f(n) = O(g(n))$*. This then lets us write* $f(n) = h(n) + O(g(n))$ *to mean* $f(n) - h(n) \ll g(n)$*, which can be interpreted as "*$f(n)$ *is approximated by* $h(n)$ *with an error of order at most* $g(n)$*." Such equations can be manipulated; multiplying both sides by a term, or adding a term to both sides, maintains the equality.*

Problem 12. *Confirm that the following argument holds:*
If $F(n) = \sum_{d|n} f(d)$ *for some arithmetic function* d*, then*

$$\sum_{n \le x} = \sum_{d \le x} f(d) \sum_{n \le x, d|n} 1 = \sum_{d \le x} f(d)[x/d] = x \sum_{d \le x} \frac{f(d)}{d} + O(\sum_{d \le x} |f(d)|)$$

Problem 13. *Use problem 12 to show that* $\tau(n) = n \cdot log(n) + O(n)$*, so the average value of* $\tau(n)$ *is approximately* $\log(n)$.

Problem 14. *If* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$*, then use problem 12 to approximate the average value of* $\sigma(n)$ *in terms of* $\zeta(2)$.