1. Arithmetic Functions

Definition. The sigma function \( \sigma(n) \) is defined as the sum of the divisors of the positive integer \( n \).

\[
\sigma(n) = \sum_{d \mid n} d
\]

The divisor function \( \tau(n) \) is defined as the number of divisors \( n \) has.

\[
\tau(n) = \sum_{d \mid n} 1
\]

The Euler totient function \( \phi(n) \) is defined as the number of relatively prime numbers less than or equal to \( n \).

The unity function \( u(n) \) is defined as \( u(n) = 1 \).

The unit function \( e(n) \) is defined as \( e(1) = 1 \), and \( e(n) = 0 \) for \( n > 1 \).

The identity function \( N(n) \) is defined as \( N(n) = n \).

The Mobius function \( \mu(n) \) is defined as \( \mu(n) = 0 \) if \( n \) is divisible by a square larger than 1, and \( \mu(n) = (-1)^k \) if \( n \) is square-free, where \( k \) is the number of prime factors of \( n \).

Definition. Given two arithmetic functions \( f, g \), we define their convolution as

\[
f \ast g(n) = \sum_{d \mid n} f(d)g(n/d)
\]

Problem 1. What is \( u \ast u(n) \)?

Problem 2. What is \( N \ast u(n) \)?
Problem 3. What function $g(n)$ has the property that $f \ast g(n) = f(n)$ for all arithmetic functions $f$ and positive integers $n$?

Problem 4. What function $g(n)$ has the property that $g \ast u(n) = e(n)$ for all positive integers $n$?

Problem 5. What function $g(n)$ has the property that $g \ast u(n) = N(n)$ for all positive integers $n$?

Problem 6. Confirm that convolution is both commutative and associative.
Problem 7. Mobius Inversion on arithmetic functions is the following statement: if $f$ and $g$ are arithmetic functions, then

$$f(n) = \sum_{d|n} g(d)$$

is true if and only if

$$g(n) = \sum_{d|n} f(d)\mu(n/d)$$

(1) Re-phrase Mobius inversion just in terms of convolution of functions, without any sums.

(2) Prove that mobius inversion holds for all functions.
2. Bounds on Arithmetic Functions

Definition. We say that $f(n) \ll g(n)$ if there are some constants $N, c$ such that for all $n > N$, $f(n) < c \cdot g(n)$.

Problem 8. Show that $\sigma(n) \ll n \cdot \log(n)$ (you can use the fact that $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$.)

Problem 9. Show that $\tau(n) \ll \sqrt{n}$ (in fact, it’s actually true that $\tau(n) \ll n^{1/k}$ for any $k > 0$, but that’s harder to prove.)

Problem 10. Disprove that $\sigma(n) \ll n$ (This means that, for any constant coefficient $c$, you need to come up with arbitrarily large counterexamples.)

Problem 11. Disprove that $\tau(n) \ll 1$. 
3. Average Values of Arithmetic Functions

**Definition.** Big Oh Notation is another way of expressing our previous comparison; if \( f(n) \ll g(n) \), then we write \( f(n) = O(g(n)) \). This then lets us write \( f(n) = h(n) + O(g(n)) \) to mean \( f(n) - h(n) \ll g(n) \), which can be interpreted as “\( f(n) \) is approximated by \( h(n) \) with an error of order at most \( g(n) \).” Such equations can be manipulated; multiplying both sides by a term, or adding a term to both sides, maintains the equality.

**Problem 12.** Confirm that the following argument holds:

If \( F(n) = \sum_{d|n} f(d) \) for some arithmetic function \( d \), then

\[
\sum_{n \leq x} = \sum_{d \leq x} f(d) \sum_{n \leq x, d|n} 1 = \sum_{d \leq x} f(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d \leq x} \frac{f(d)}{d} + O\left( \sum_{d \leq x} |f(d)| \right)
\]

**Problem 13.** Use problem 12 to show that \( \tau(n) = n \cdot \log(n) + O(n) \), so the average value of \( \tau(n) \) is approximately \( \log(n) \).

**Problem 14.** If \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \), then use problem 12 to approximate the average value of \( \sigma(n) \) in terms of \( \zeta(2) \).