Lesson 4 Problem 4 Solution

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Problem 4.
We will optimize the solution by solving all the parts at the same time. First of all, let me show a parallelogram has all the properties outlined. Indeed, suppose $AB \parallel CD$ and $AD \parallel CB$. Then $\angle ABD = \angle BDC$, $\angle BAC = \angle DCA$, $\angle DAC = \angle ACB$ and $\angle ADB = \angle DBC$. Then the SAS test tells us that $\triangle ADC = \triangle CBA$, which implies $AB = CD$ and $AD = BC$. This is parts a) and b). Let O be the intersection of diagonals. Then by the SAS test we know have $\triangle AOB = \triangle COD$, which implies part c). Now let us do the other directions. **a)** Suppose $AB = CD$ and $AD = BC$. Then by SSS $\triangle ADC = \triangle CBA$, which implies $\angle BAC = \angle DCA$ and $\angle DAC = \angle ACB$ which in turn imply $AB \parallel CD$ and $AD \parallel CB$.

b) $AB \parallel CD$ implies that $\angle DCA = \angle CAB$ and $\angle BDC = \angle DBA$. Then together with $AB = CD$ we get $\triangle AOB = \triangle COD$, which implies $BO = DO$ and $AO = OC$. Then we can conclude that $ABCD$ is a parallelogram by part c), which is proved below.

c) Suppose $BO = DO$ and $AO = OC$. Then by SAS we have $\triangle AOB = \triangle COD$ and $\triangle AOD = \triangle COB$, which imply $AB = CD$ and $AD = BC$, and we are done by part a)

d) Suppose $AC \perp BD$. Then in $\triangle ABD$ we get that the median AO coincides with the altitude. Then $AD = AB$, implying that all the sides are equal. Conversely, if all the sides are equal, then the median must coincide with the altitude, so the diagonals are perpendicular.