NEGATIONS

BEGINNER CIRCLE 10/27/2012

1. WARM UP: OPPOSITE DAY

Today is Opposite day. Isaac and Jeff decide to be very clever and hold a conversation only saying the exact opposite of what they mean. Can you translate what they are saying without just adding the word "not" in front of the sentence? For example, if Jeff said "All swans are white" then what he really means is "There exists a swan that is not white"

Isaac: My day is anything but good.

My day is good.

Jeff: There is a cloud outside!

There are no clouds outside.

Isaac: I have done only 1 problem on my homework!

I have done more or less than 1 problem on my homework.

Jeff: I have done all of the problems on my homework.

I have not finished my homework.

Isaac: I think that outside is not 75 degrees.

I think it's 75 degrees outside.
Jeff: Nobody will show up to math circle today.

Isaac: Way! I agree. There will be at least one person that is missing.

You are wrong. Everyone will show up to math circle.

Jeff: If not a single person shows up, I will teach!

\[ n \rightarrow p \quad \text{If no one shows up, I won't teach,} \]

\[ n \rightarrow !p \]

Isaac: If today is Tuesday, I will go to the store!

\[ \text{If today is Tuesday, I won't go to the store} \]

Now have a conversation with your neighbor!
2. General Statements

Let us talk about general statements. A general statement is a fact about a lot of objects: for instance

“Every Math Circle instructor is over 4 feet tall”

is a statement about all Math circle instructors.

Problem 1. Suppose we found a person that was only 3 feet tall. Then what could we say about them?

They are not a math circle instructor.

A different type of general statement tells us about the existence of certain things, like

“Every Math Circle classroom contains an instructor”

This statement says that we can always find a certain type of object within a set.

Problem 2. Suppose we found a classroom without an instructor. What could we say about the classroom?

It is NOT a Math Circle classroom.

Sometimes the wording of a statement can be tricky, especially with the word or. Take for instance, the statement,

“In order to be a Math Circle instructor, you must like teaching or be handsome.”

However, this statement doesn’t mean that all of our Math Circle instructors either like teaching or are handsome: take for instance Derek, who is both of these things. When we write “A or B”, we mean “A or B or both”.
3. Negations

What do we mean when we say the opposite of a statement? Let us look at an example of a negation:

"A = I like every kind of pie."

How do we properly negate this sentence?

"(not A) = There is a kind of pie that I don't like."

A negation of a statement is true if and only if the original statement is false. If we take a negation of a negation, we should get our original statement back. For instance

"(not not A) = There is no kind of pie that I don't like"

is equivalent to the statement that “I Like every kind of pie”

Problem 3. Negations can be a bit tricky to figure out! Sometimes, we may have a statement that looks like a negation, but really isn’t. For instance, look at this statements:

"B = I like no kind of pie"

Why is this not a negation of “I like every kind of pie”

If I like apple pie, but hate cherry, then the statements are both false at the same time.

Problem 4. Write down a negation for the following statements:

(1) Derek’s favorite color is Blue

Derek’s favorite color is not blue

(2) Isaac likes every color except red.

Isaac does not like a color that is not red.

(3) Every number is even.

There is a non-even number
(4) There is a letter of the alphabet written with just one pen stroke.

All letters of the alphabet require more than 1 pen stroke.

(5) In order for it to be hot outside, it can be sunny or it can be summer.

It is hot outside when it is not sunny nor summer.

**Problem 5.** Match each statement to the one which is the proper negation of it

- Every Turtle is slow
- There is a fast turtle
- Every Turtle is either fast or slow
- There is a turtle that is fast and slow
- Every turtle is not fast
- There are turtles that are not slow and not fast
- There exists a turtle that is not slow
- All turtles are either not fast, or not slow
Problem 6. Let us look at general statements of the type

"Every A has property B"

These are statements like

Every instructor in Math Circle has a Beard

Compare this to the statement

There is a Math Circle instructor that has no beard

(a) Explain why if the first statement is true, then the second statement is false.

For (1) to be true, **ALL** math Circle instructors have beards, so none of them have no beards.

(b) Explain why if the second statement is true, the first statement is false.

If one Math Circle instructor does not have a beard, then not all have a beard.

(c) Conclude that the first statement is the negation of the second statement.

Since they are only true when the other is false, they are negation.

Problem 7. Using the previous problem as a hint, explain why the opposite of "Every A has property B" is "There is a A that does not have property B"

Any A without B instantly invalidates that all A have B.
4. CONTRAPOSITIVE

Consider the statement

"If it is raining outside, then Derek will open his an umbrella."

This is a "if A, then B" type of statement. The rule of contrapositive says that this is the same thing as "if not B, then not A." How can we rephrase this top sentence?

"If Derek is not holding an umbrella, then it is not raining."

We call this reversed statement a contrapositive. A good way to remember contrapositive is with a picture

\[
\begin{array}{c}
A \quad \rightarrow \quad B \\
\text{not } B \quad \leftrightarrow \quad \text{not } A
\end{array}
\]

**Problem 8.** Explain why the statement "If you are Math Circle instructor, then you are older than 18" is logically equivalent to "If you are not yet 18, you are not a Math Circle instructor"

Since being 18 or older holds for any Math Circle instructors, then you are definitely not a MC instructor if you aren't 18.

**Problem 9.** Explain why "If A is true, then B is true" is logically equivalent to "If B not true, then A is not true"

Since B holds for any A, then any \(\not\) not B cannot be A.

**Problem 10.** Can you take the contrapositive of the following statements?

1. If it is sunny, then it is not raining

   If it is raining, it is not sunny.

2. If every swan is white, then no swan is black

   If there is a black swan, then not every swan is white.
(3) If it is too late, then you should go to sleep
   You should stay up if it is not too early.

(4) If you don’t eat food, then you’ll become hungry.
   You If you aren’t hungry, then you ate food.

Proving the contrapositive of a statement can sometimes be easier than proving the statement itself. For instance let us try to prove
   “If $x^2$ is even, then $x$ is even”

**Problem 11.** (A proof using the contrapositive)

(a) What is the contrapositive of the statement “If $x^2$ is even, then $x$ is even”
   If $x$ is not even, then $x^2$ is odd.

(b) Use mod 2 arithmetic to prove that the square of an odd number is odd.
   $$x \text{ odd} \quad x \mod 2 = 1 \quad 1^2 = 1 = \text{odd number}$$

(c) Conclude that if $x^2$ is even, then $x$ is even.
   **Contrapositive is equivalent to original proof,**
5. **Negation of And and Or**

How do we negate a statement like

“Today is Cold and Rainy”

To help us understand this statement, we use a **truth table**.

**Problem 12.** Fill out the truth table

<table>
<thead>
<tr>
<th>Today is Cold</th>
<th>Today is Rainy</th>
<th>Is Today is Cold and Rainy?</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

What should the negation of this statement be? Well, let us look:

**Problem 13.** Fill out the truth table

<table>
<thead>
<tr>
<th>Today is Cold</th>
<th>Today is Rainy</th>
<th>It is not the case that Today is Cold and Rainy?</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Problem 14.** Finish filling out these truth tables:

<table>
<thead>
<tr>
<th>Today is Cold</th>
<th>Today is Rainy</th>
<th>It is not Cold and it is not Rainy</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Today is Cold</th>
<th>Today is Rainy</th>
<th>It is not Cold or it is not Rainy</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Why do these tables show that the proper negation of

“*A and B*”

is the statement

“*Not A or Not B*”

since when either is **T or F**, the other is the **opposite**.
6. Proof by Contradiction 1

Sometimes in order to prove something, it is easier to prove that it cannot be false. Why wouldn't something be false? The best way to see this is by example. Let us prove that the square root of 2 is irrational.

Recall that a number is called irrational if it is not expressible as a fraction. How will we prove that $\sqrt{2}$ is not rational? We will assume that $\frac{a}{b}$ expressible as a fraction, and prove that it can't be expressible as a fraction. Since it can't be true that $\sqrt{2}$ is both a fraction and not a fraction, our original assumption must be incorrect.

**Problem 15.** (Proof that $\sqrt{2}$ is irrational)

(a) Assume (for our contradiction) that $\sqrt{2} = \frac{a}{b}$. Why is it that we can choose $a$ and $b$ so that only one of them is divisible by 2?

Because keep dividing by 2 to numerator and denominator until one is odd, but can't both be.

(b) We then have that $2 = \frac{a^2}{b^2}$. Can you prove that $a$ is even?

If $a = 2m+1$, then $a^2 = 2k+1$ and $(\frac{2m+1}{2n+1})^2 \neq 2$ is not even, so $b^2$ dividing into $a^2$ can only give an odd number $a = 2m$, so $a^2 = 4m^2$ and 4 divides $a^2$.

(c) Why does 4 divide $a^2$? $a^2$ can only give an odd number $a = 2m$, so $a^2 = 4m^2$ and 4 divides $a^2$.

(d) If 4 divides $a^2$, how do we know that 2 divides $b^2$?

\[
\frac{a^2}{b^2} = \frac{4m^2}{b^2} = 2\quad \text{so } b^2 = 2m^2 \text{ and } b^2 \text{ is even}
\]

(e) If 2 divides $b^2$, is $b$ even or odd?

\[b \text{ is even}\]

(f) Why is this a contradiction? Conclude that there is no $a$ and $b$ so that $\frac{a}{b} = \sqrt{2}$.

We found both $a$ and $b$ to be even when we specified that one is odd, which means original premise is wrong.

**Problem 16.** Why doesn't this proof work when we try to show that the square root of 4 is not rational?

Since $\sqrt{4} = 2$ is rational
Problem 17. Can you use the same kind of proof to show that the square root of 3 is not rational?

Assume \( a \) and \( b \) both odd since otherwise \( a^2 \) is even and cannot bring about 3.

\[
\frac{a^2}{b^2} = 3 \quad \text{so both } a^2, b^2 \text{ are odd.}
\]

Now \( a^2 = 3b^2 \), let us test two odd numbers as \( a = 2m+1, b = 2k+1 \)

\[
(2m+1)^2 = 3(2k+1)^2
\]

\[
4m^2 + 4m + 1 = 12k^2 + 12k + 3
\]

\[
-1 \quad -1
\]

\[
4m^2 + 4m = 12k^2 + 12k + 2 \quad \div 2 \quad \div 2
\]

\[
2m^2 + 2m = 6k^2 + 6k + 1
\]

even \quad odd \quad CONTRADICTION

so \( \sqrt{3} \) is not rational.
7. Proof by Contradiction 2

One of the classic proofs of mathematics is that there is an infinite number of prime numbers. While it is hard to show that there is an infinite number of primes, it is easy to show that it is not possible for there to be a finite number of primes. We will use a proof by contradiction to show that this is true.

**Problem 18.** (Proof of infinitely many primes)
(a) If we want to prove that there are infinitely many primes, what should we instead assume for our contradiction?

Assume a finite number of primes.

(b) With our assumption, why is it that we can list all of the primes,

\[ p_1, p_2, p_3 \ldots p_n \]

Because there are only finite primes,

(c) Why is it that \( p_1 \) does not divide \( p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1 \). (Hint: look at the remainder)

\[ p_1 \text{ divides it by } p_2 \times p_3 \times \ldots \times p_n \text{ remainder } 1 \]

since \( p_1 \neq 1 \).

(d) Why is it that none of the \( p_1, p_2, \ldots, p_n \) divide \( p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1 \)?
The same reasoning

(e) What can we conclude about \( p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1 \)? How is this a contradiction?

It is prime!!! We assumed we had them all.

**Problem 19.** Suppose that we know that \( p_1, p_2, p_3 \ldots p_n \) are prime numbers. Does this prove that \( p_1 \times p_2 \times p_3 \times \ldots p_n + 1 \) is a prime number? Why or why not?

No, since \( 3 \times 5 + 1 = 16 \).
8. PARADOXES

Sometimes it can be hard to determine if a statement is true or false. In fact, sometimes, it is impossible to tell if a statement is true or false. Take for instance the statement

"This statement is false."

This seems troublesome. Why?

If it’s false, then the statement is true, but if it’s true, then the statement is false.

Here is a problem (from Clint) that is particularly interesting:

**Problem 20.** Sam is very pleased with his most recent purchase, a shiny book called the *Complete Non-Self-Reference Reference* (CNSRR). The ads tell us that this book mentions exactly all the books that doesn’t mention themselves. For instance, *Harry Potter and the Sorcerer’s Stone* doesn’t mention itself, as a book, anywhere in the text, so CNSRR mentions it. On the other hand, the dictionary does mention itself, so it does not get mentioned in CNSRR. When Edanel hears about Sam’s new purchase, he gets worried. “I think there’s a problem with your new book,” he says. What does Edanel mean? What’s the problem?

Does it contain *ITSELF*/?!
Problem 21. Pablo, Gary, Sara, Mark, and Wanda were playing Clue last week. Each one was playing as one of five characters: Professor Plum, Mr. Green, Miss. Scarlet, Colonel Mustard or Mrs. White. Everybody but one person committed a crime in a certain room with a certain weapon. You interview the suspects, but quickly realize that they only tell lies. Using these clues, can you solve the mystery?

Interview Transcript

(1) Pablo: There is a person who is not one of the following
   (a) Pablo
   (b) The woman who is playing Colonel Mustard
   (c) The Man who used Wrench
   (d) the Kitchen criminal
   (e) The one who is innocent.

(2) Gary: At least one player’s name did start with the same letter of the character they were playing. No player’s first name is the same letter of character.

(3) Sara: Wanda did not steal the knife.

(4) Mark: The person who isn’t Gary or the person who isn’t Sara were not in the Ballroom and not in Study

(5) Wanda: Mrs. White did not use the Lead Pipe or Professor Plum did not use the Candlestick

(6) Pablo: The murderer did not have the rope

(7) Gary: Mark was not seen in the Cellar with the Candlestick

(8) Sara: Whoever broke the wrench did not do it in the Ballroom.

Wanda → knife ← kitchen &
White ← Lead Pipe
Plum ← Candlestick ← Mark ← Cellar
Wrench ↔ ballroom ↔ Gary
rope ↔ murderer
Sara ↔ Mustard ↔ Study