Alternative solution to L4.3

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The solution above is the geometric interpretation of the algebraic solution you already have – perhaps some of you will find this interpretation more clear.

**Problem 3.**
We will give the solution straight for part b) of the problem. First suppose \( b > 0 \). Then let us mark the points \( 0, b, 2b, 3b, \ldots \) on the coordinate line. And the same for negative multiples of \( b: -b, -2b, -3b \) and so on.

If we now represent \( a \) as a point on the coordinate line, it will fall between some pair of marked points, lets call them \( qb \) and \((q + 1)b\).

If \( a \) falls directly on a marked point, we will call that point \( qb \):

Now let us set \( r = a - qb \). Since \( a \) is between \( qb \) and \( qb + b \) we know that \( r = a - qb < qb + b - qb = b \), so \( r < b \) and clearly \( r \geq 0 \). Thus these \( r \) and \( q \)
work, which concludes the case $b > 0$.

If $b < 0$, we can use the previous case: if we find $q$ and $0 \leq r < |b|$ such that $a = q(-b) + r$, then it also holds that $a = (-q)b + r$ which is the desired formula.

The argument we presented proves that $q, b$ exist. As far as the uniqueness goes, one can either follow the argument in part a) of the original algebraic solution, or consider the following geometric viewpoint: if one chooses $q'b$ to be any point with $q' > q$ where $q$ is the one we chose, then $q'b$ will be to the right of $a$ and $r' = a - q'b$ will have to be negative. If we choose $q' < q$, then the point $q'b$ will be at least length $b$ far from $a$ to the left, and so $r' = a - q'b > b$ which is also prohibited. So the choices of $q$ and $r$ we made were in fact forced, and thus unique. As for the case $b < 0$, uniqueness follows from the uniqueness of the remainder when divided by $-b$: if $q, r$ are unique solutions for the equation $a = q(-b) + r$ with $0 \leq r < b$, then the ones for $a = qb + r$ are unique as well since they differ only by changing the sign of $q$. 