Lesson 4 Problem 3 Solution.

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Problem 3
a) (Existence)
Solution 1: Consider the numbers: 0, b, $2b$, $3b$... After some point, all numbers on the list will be greater than a. For example, ab will be greater than a, and so will all the numbers that follow ab. Let q be the biggest number such that $qb \leq a$. Now we only have to show $a - qb < b$. Suppose that is not true, $a - qb \geq b$. Then $a \geq qb + b = (q+1)b$. This is a contradiction to q being the biggest number such that $qb \leq a$. We can conclude that there exists q and $r = a - qb$ such that $a = qb + r$ with $r < b$.

Solution 2: Let q be the integer part of a/b in decimal. For example if $a/b = 7.6666...$, then $q = 7$. (This can be denoted as $q = \lfloor a/b \rfloor$). Then

$$\frac{a}{b} - q < 1$$

multiplying by $b > 0$ on both sides we get $a - qb < b$, which lets us set $r = a - qb$ and be done.

(Uniqueness) Suppose there is another pair q', r' satisfying the condition. So $a = qb + r$ and $a = q'b + r'$. Subtract one from the other, and we get

$$0 = b(q - q') + r - r' \tag{1}$$

$$b(q - q') = r' - r \tag{2}$$

If $q = q'$ we must also have $r = r'$ by equation (1), which means the pairs wee actually the same. If q and q' are distinct integers, $|b(q - q')| \geq b$. But since $0 \leq r < b$ and $0 \leq r' < b$ we have $|r - r'| \leq b - 1$. Therefore equation (2) cannot hold. Contradiction, so q and r must be unique.

b) To deal with the situation when a and b could be negative, we consider three separate cases:

1) Both a and b are negative. Then we can apply part a) to get

$$(-a) = (-b)q + r$$

for some $0 \leq r < |b|$. Multiplying both sides by -1 we get

$$a = bq - r$$

If $r = 0$, then this is already in the required form. Otherwise, we can also write

$$a = b(q-1) - r - b = b(q-1) + (-b-r)$$

Since b is negative and $1 \leq r < |b|$ we have $0 < -b - r < |b|$ and so

$$a = b(q-1) + (-b-r)$$

is the required form.

2) $a > 0$ and $b < 0$. Then we can apply part a) to get

$$a = (-b)q + r$$

for some $0 \leq r < |b|$. This can be rewritten as

$$a = b(-q) + r$$

which concludes this case.

3) $a < 0$ and $b > 0$. Then we can apply part a) to get

$$-a = bq + r$$

for some $0 \leq r < |b|$. This can be rewritten as

$$a = b(-q) - r$$

If $r = 0$, then this is already in the required form. Otherwise, we can also write

$$a = b(-q-1) - r + b = b(-q-1) + (b-r)$$

Since b is positive and $1 \leq r < |b|$ we have $0 < b - r < |b|$ and so

$$a = b(-q-1) + (b-r)$$

is the required form.