Infinity III

Your room is the last one on the right

Math Circle

November 5, 2017

1. Today we are going to talk about some infinite Paradoxes. Before we do, it's important that we talk about the difference between a paradox and a contradiction. Most people refer to these two things interchangeably, but they aren't quite the same.

   A paradox is a statement that makes sense logically, but apparently violates intuition of common sense.

   A contradiction is a statement that is logically false, and may or may not make sense intuitively.

   Out of the following which are paradoxes, which are contradictions? Why?

   (a) Here is a proof that $1 = 2$. Suppose that we have two numbers a and b which are equal, see that:

   $$a = b$$
   $$a^2 = ab$$
   $$a^2 - b^2 = ab - b^2$$
   $$(a - b)(a + b) = b(a - b)$$
   $$a + b = b$$

   therefore if we select $a = b = 1$, we have that $2 = 1$

(b) Do you know the number i? Mathematicians define $i = \sqrt{-1}$. Let's do some math with i.

$$\begin{aligned} -1 &= i^2 \\ &= \sqrt{-1}\sqrt{-1} \\ &= \sqrt{(-1)(-1)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

hence $1 = -1$.

(c) In between every 2 distinct rational numbers there are infinitely many real numbers. In between every 2 distinct real numbers there are infinitely many rational numbers. Nevertheless there are more real numbers than rational numbers.

(d) Nothing is better than eternal happiness.
A ham sandwich is better than nothing.
Therefore a ham sandwich is better than eternal happiness.

(e) Let $C = 1/2 + 1/4 + 1/8 + \dots$:

$$\begin{aligned} C &= 1/2 + 1/4 + 1/8 + \dots \\ 2C &= 1 + 1/2 + 1/4 + 1/8 + \dots \\ 2C &= 1 + C \\ C &= 1 \end{aligned}$$

(f) Let $C = 1 - 1 + 1 - 1 + 1 - \ldots$:

$$\begin{aligned} C &= 1 - 1 + 1 - 1 + 1 - \ldots \\ C - 1 &= -1 + 1 - 1 + 1 - 1 + 1 - \ldots \\ C - 1 &= -C \\ C &= 1/2 \end{aligned}$$

Time for our video! Today, we are watching another video by Numberphile. This on is simply called 'Infinity Paradoxes' and brings up 4 interesting paradoxes that occur when you think about the infinite.

`https://www.youtube.com/watch?v=dDl7g_2x74Q`

Ok, let's look at these four paradoxes more closely. First up, Hilbert's Hotel. David Hilbert was an extremely influential mathematician around the turn of the 19th century, and this hotel is of his design.

(a) First, it is said that Hilbert's Hotel has an infinite number of rooms labeled $1, 2, 3, \ldots$. What is the cardinality of the set of all of the rooms in Hilbert's Hotel?

(b) Now, initially the room starts off as full. What is meant mathematically by the hotel being full? Be specific!

(c) The video shows how if you had 1 more guest show up, then they could be given a room by moving every single person down 1 room. By doing so, they show that there is a bijection between $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$. Can you explain this connection between the hotel and the bijections?

(d) The video also describes how you if you had a bus with infinitely many people in it, you could make room for all of them in your hotel. How?

(e) Let's say that the patrons of Hilbert's Hotel are tired of moving, and they pressure the Governor to pass a new law. This new laws say that in order to make room for new guests in Hilbert's Hotel, you can't move more than %10 of the patrons. What is the maximum number of new guests that you can still accept?

2. Paradox number 2, Gabriel's Horn. In one of his videos (Supertasks), Michael form vscauce talks about something he calls Gabriel's cake. Let's study that thing, because I think that it is a little more clear.

   (a) To construct Gabriel's cake, you first take a cube shaped cake, and cut it in half length wise. Put one half of the cake on a serving tray, and keep the other half in the cutting board. Then take this half and cut it in half lengthwise again. Move one of the two quarters onto the cake already on the serving tray. Do this an infinite number of times. The product that you get at the end is called Gabriel's cake. Show that Gabriel's cake has finite volume.

(b) How many times did I have to cut the cake? What is the cardinality of set of cuts that I made?

(c) If the initial cube cake is 10 by 10 by 10 inches, then what's the initial surface area of the cake?

(d) What is the surface area after 1 cut? What about after 2 cuts?

(e) An infinite sum is something of the form $a_1+a_2+a_3+\ldots$... which goes on forever. Mathematicians say that an infinite sum equals infinity if for any number that you think of (let's call it M) I can find a natural number (an index) m such that all of the finite sums $a_1+\cdots+a_m$ and $a_1+\cdots+a_{m+1}$ and $a_1+\cdots+a_{m+2}$, and on and on are all greater than M. Prove that $1+1+1+\cdots=\infty$.

(f) Prove that $1+\frac{1}{4}+\frac{1}{9}+\cdots\neq\infty$.

(g) Prove that $1+\frac{1}{2}+\frac{1}{3}+\cdots=\infty$.

(h) Prove that the surface area of Gabriel's cake is infinity in this way.

3. The next paradox is a paradox of probability. When you throw a dart at a dartboard there is a probability 0 that it will hit any particular point. Yet, it has to hit somewhere!

   (a) Suppose that the entire dart board has an idea of 1 meter squared, and you throw the dart so that it is just as likely to hit any one point as any other point. What are the chances that you'll hit a section of the board which is 1/2 meter squared?

   (b) What are the chances that you'll hit a region with surface area A?

(c) What's the area of a single point?

(d) What are the chances that you'll hit a single point?

(e) If you consider the sum of the chances of hitting every single point on the board you would get an infinite sum of the form $0 + 0 + 0 + \ldots$. This infinite sum is equal to zero. That means that your chances of hitting anywhere on the board must be 0. Is the previous argument a paradox, or a contradiction? Why?

4. Alright, last one. Let's recall the final paradox, where you play this double or nothing game.

   (a) Mathematicians say that the expected value $\mathbb{E}[\cdot]$ of a random event x is the sum of the probability of all outcomes times their payoff respectively. Let's say that I'll roll a fair die, and I'll give you \$1 if it's a one, \$2 if it's a two, etc. What is the expected value of your payoff from this game?

   (b) Suppose that I change the rules of the game so that if the roll is prime I give you \$1, and if it's not, I'll take \$1 from you. What is the expected payout of this game?

(c) Mathematicians say that a game is 'in your favor' (and so you should play as many times as you are able) if your expected returns from playing that game is > 0. Why do we say this? Why is the expectation important?

(d) Prove that the expected value of the infinite double or nothing game is infinite.

(e) Prove that if you have to pay an amount M to play the double or nothing game, the game is in your favor. No matter what M is.

(f) How much would you pay to play this game? Be honest!