The Missing $1  Three people rent a room for the night for a total of $30. They each pay $10 and go upstairs. The owner then realizes the room was only supposed to be $25. He sends up the bellhop to return the $5. He can’t split it evenly, so each person takes $1, and they give the other $2 back to the bellhop. Now, each person has paid $9 ($27 total) and the bellhop has $2. What happened to the other dollar?
One Bad Coin You have 12 coins. One coin weighs slightly more or slightly less than the others, but you do not know which coin or whether it is heavier or lighter than the rest. Your only tool is a balance scale, as shown below. One weighing consists of placing any number of coins on either side of the scale, and then weighing to see which side is heavier. Using only three weighings, how can you determine which coin is bad, and whether it is heavier or lighter than the rest?
The Milk Man  A milk man has two full 10-gallon containers of milk. He also has an empty 3-gallon container, and an empty 5-gallon container. He has two customers who want two gallons of milk each. He has no spare containers, and no way to mark his containers. He also does not want to waste any milk by pouring it out. How does he make the sale?
The Big Number  A teacher writes a large number on the board, and asks her class to list the number’s divisors. The 1st student said "2 is a divisor." The 2nd student said "3 is a divisor." This pattern continued, until the 30th student said "31 is a divisor." The teacher then tells the class that exactly two students spoke wrongly, and that those students spoke consecutively (in a row). Which two students spoke wrongly?
**Capri Sun vs. Kool Aid** You have two glasses of liquid, Glass 1 is a 100ml glass of Capri Sun and Glass 2 is a 100ml glass of Kool Aid. First, you take a spoonful of Glass 1 (Capri Sun) and mix it into Glass 2 (Kool Aid). Then, you take a spoonful of Glass 2 (which is now a mixture) and mix it into Glass 1. Which glass now contains the *least* amount of the other drink?
Cars in a Desert  A very important mathematician needs to get across the desert. But, the only type of car available can only hold enough fuel to get halfway across the desert. Luckily, the mathematician is very wealthy and can afford many of these cars, and many men to drive them. The cars can transfer fuel from one to the other while driving across the desert. How can we get the mathematician across the desert, and how many cars do we need?
Challenge: NIM

NIM is a simple, two-player game. The game starts with some number of objects in some number of piles on the table. We call the player who goes first Player 1, and the player who goes second Player 2. On a player’s turn, they can take any number of objects from any one pile of their choice. The player who is forced to pick up the last match is the loser. If you’d like to see a demonstration, call over an assistant and we can play a game.

(a) Play a game of NIM with just two piles. Try different numbers for the size of each pile, and pay attention to who wins, Player 1 or Player 2. In particular, try a game with two piles where one pile only has one card in it. Who wins?

(b) Who wins when both piles start with the same number of cards? When they start with different numbers?

(c) The secret of this game lies in binary numbers. Remember, we can convert a number into binary by splitting it into powers of 2. For example,

$$43 = (1 \times 32) + (0 \times 16) + (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1).$$

So, if we wanted to write 43 in binary, we would write

$$43_{10} = 101011_2$$

Try converting the following numbers into binary: 10, 25, 64.

(d) Now we’ll learn about a special type of addition called the NIM-sum. To perform the NIM-sum of two binary numbers, we add them digit by digit, and then turn any even digit into 0 and any odd digit into 1. For example, let’s compute the NIM-sum of 43 and 17. In binary, $43_{10} = 101011_2$ and $17_{10} = 010001_2$. When we add them digit by digit, we get $111012$. Once we turn all the even digits into 0 and the odd digits into 1, we get $111010$ as our NIM-sum. We’ll explain what this means in part (e). For now, practice computing the following NIM-sums:

(i) $10 \oplus 10$
(ii) $25 \oplus 64$
(iii) $10 \oplus 25 \oplus 64$

(e) Now for the fun part! You should have found in part (b) that when there are two piles with the same number of cards, Player 1 loses the game (ask for help if you still don’t understand why). You may have also noticed in part (d) that any number NIM-summed with itself gives 0 as the result. This is not a coincidence! It turns out that any time the NIM-sum of the piles is equal to 0, the game is in a “winning position.” You don’t have to prove this, but it turns out from any non-winning position, you can make one move to get to a winning position. Also, any move from a winning position will get you to a non-winning position.
**BIG QUESTION:** Is our game even worth playing? If the game begins in a winning position, who is guaranteed to win?

(f) Play a couple games of NIM with more than two piles. Practice computing the NIM-sum of the game, and try to figure out what move to make in order to create a winning-position. If you’re feeling bold, challenge Eli to a game of NIM for a prize.