INVERSE FUNCTIONS

INTERMEDIATE GROUP - MAY 14, 2017

Cantor’s Continuum Hypothesis

The proposal made by Georg Cantor in the 19th century states that there is no infinite set with a cardinality larger than that of the set of natural numbers, \( \mathbb{N}_0 \) and smaller than the set of real numbers, \( \mathbb{N}_1 \).

We now know that with the standard axioms of set theory today, we can neither prove nor disprove the hypothesis.

Warm Up

We say that \( a \) is the modulo-\( m \) residue of \( n \) when \( n \equiv a \pmod{m} \), and \( 0 \leq a < m \).

The set of congruence classes of modulo-\( m \) is the set of all modulo-\( m \) residues. We typically write \( \mathbb{Z}_m \) to denote the set of congruence classes modulo-\( m \).

(1) What is the modulo-6 residue of 23?

(2) What is the modulo-7 residue of 16?

(3) List all the elements of \( \mathbb{Z}_6 \).

(4) List all the elements of \( \mathbb{Z}_7 \).
Inverse Functions

Bijective functions can be inverted. For a given function \( f : X \rightarrow Y \), the inverse function reverses the roles of inputs and outputs. It is denoted by \( f^{-1} : Y \rightarrow X \) and has the property that \( f^{-1}(y) = x \) if \( f(x) = y \).

**Example 1.** Consider the bijective function \( f : \mathbb{Z} \rightarrow \mathbb{Z}_{\text{even}} \) given by
\[
f(n) = 2n.
\]
The inverse function of \( f \) is given by \( f^{-1} : \mathbb{Z}_{\text{even}} \rightarrow \mathbb{Z} \) where
\[
f^{-1}(n) = \frac{n}{2}.
\]

**Problem 1.** Why is the bijectivity of the original function a necessary condition for it to have an inverse?

1. Why must \( f \) to be onto in order to be invertible?

2. Why must \( f \) to be one-to-one in order to be invertible?
Problem 2. Consider the function $f$ defined by the table below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

(1) Find $f^{-1}(-2)$.

(2) Find $f^{-1}(3)$.

(3) Can you compute $f^{-1}(10)$? Explain why or why not.

(4) Define the inverse of $f$ in the table below. What do you notice about the columns of the table for $f$ compared to the columns of the table for $f^{-1}$?

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{-1}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
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<tr>
<td>5</td>
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</tbody>
</table>
Problem 3. On the \( xy \)-plane below, draw the function \( y = x \). Then draw the following functions and their inverses.

- \( y = 2x \) and \( y^{-1} = \frac{1}{2}x \)
- \( y = \frac{1}{2}x + 1 \) and \( y^{-1} = 2x - 2 \)

What do you notice about the graphs of the functions and their inverses?

Problem 4. Using your observations from Problem 3, draw the inverse of the function shown in the graph below in the same \( xy \)-plane.
Problem 5. Let \( f(x) : \mathbb{R} \to \mathbb{R} \) be given by
\[
f(x) = 3x - \frac{1}{2}.
\]
To find \( f^{-1}(x) \):

1. Solve \( y = 3x - \frac{1}{2} \) for \( x \) in terms of \( y \).

2. Rename the input and output by switching the variables \( x \) and \( y \). Write down the formula for the new function.
Problem 6. Let \( \mathbb{R}_{\geq 0} \) be the set of non-negative real numbers. Let \( \mathbb{R}_{\geq 1} \) be the set of real numbers greater than or equal to one. Let \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 1} \) be given by \( f(x) = \sqrt{x} + 1 \).

(1) Pick at least five possible values of \( x \in \mathbb{R}_{\geq 0} \) and find their corresponding values of \( f(x) \). Write your results in the table below.
Hint: Use the values of \( x \) for which \( \sqrt{x} \) is an integer.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
</table>

(2) Plot the graph of the function the \( xy \)-coordinates below by using the points you found from part (1).
(3) Prove that \( f(x) \) is a bijection.

(4) Finding a formula for \( f^{-1}(x) \):
   (a) Solve \( y = \sqrt{x} + 1 \) for \( x \) in terms of \( y \).

   (b) Rename the input and output by switching the variables \( x \) and \( y \). Write down the formula for the new function.
Problem 7. Consider the function

\[ f(x) = \frac{x + 1}{x}. \]

(1) Find the domain of \( f \).

(2) Find the range of \( f \) by first rewriting \( f \) as a mixed number.

(3) Find \( f^{-1} \).

Problem 8. Is there a function \( f : \mathbb{R} \to \mathbb{N} \) which is invertible? If so, give an example. If not, explain why no such function can exist. How about a function \( g : \mathbb{Q} \to \mathbb{N} \)?
**Problem 9.** Suppose that Lenny knows the function \( f : \mathbb{N} \to \mathbb{Z} \) and its inverse \( f^{-1} : \mathbb{Z} \to \mathbb{N} \). She picks a number \( n \in \mathbb{N} \) and finds that \( f(n) = z \) for some \( z \in \mathbb{Z} \). What will she get if she calculates \( f^{-1}(z) \)?

**Problem 10.** Recall that a set \( X \) is an **infinite set** if a proper subset \( Y \) of \( X \) is isomorphic to \( X \). That is, there is a bijection \( f : X \to Y \), where \( Y \) is a proper subset of \( X \). Prove that finding a bijection \( g : Y \to X \) would also show that \( X \) is an infinite set.
Problem 11. (Challenge) Consider the set $\mathbb{Z}_5 = \{\text{congruence classes of integers mod } 5\}$

(1) List the elements of $\mathbb{Z}_5$.

(2) Consider the function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$, given by $f(x) \equiv 3x \pmod{5}$. Fill out the table representing the function below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
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</tbody>
</table>

(3) Explain why $f(x)$ is a bijection.

(4) Find the table of values for $f^{-1}(x)$. Can you express $f^{-1}(x)$ in the form $f^{-1}(x) = kx$ for some $k \in \mathbb{Z}_5$?

<table>
<thead>
<tr>
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<th>$f^{-1}(x)$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
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<td>2</td>
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<td>4</td>
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</tr>
</tbody>
</table>
(5) Finding the inverse function using multiplicative inverses:
   (a) Find the multiplicative inverse of 3 in mod 5. That is, find a number \( n \in \mathbb{Z}_5 \) such that \( 3 \cdot n \equiv n \cdot 3 \equiv 1 \) (mod 5).

   (b) Therefore, in \( \mathbb{Z}_5 \),
   \[
   3^{-1} = \underline{\hspace{2cm}}
   \]

   (c) Use this to find the inverse of the function \( y = 3x \) in \( \mathbb{Z}_5 \). Does this agree with the answer you go before by reversing the table?

(6) Find the inverse of \( g : \mathbb{Z}_5 \to \mathbb{Z}_5 \) where \( g(x) \equiv 5x - 1 \) (mod 5).
(7) Prove that any function $h : \mathbb{Z}_5 \to \mathbb{Z}_5$ of the form $h(x) \equiv ax + b \pmod{5}$ where $a \neq 0$ has an inverse.

Hint: Start by showing that for every $a \in \mathbb{Z}_5$ where $a \neq 0$, there is an $a^{-1}$ such that $a^{-1} \cdot a \equiv a \cdot a^{-1} \equiv 1 \pmod{5}$. 