BIJECTIONS AND INFINITE SETS
INTERMEDIATE GROUP - APRIL 23, 2017

Warm Up

(1) Recall the following definitions from last week:

(a) A function $f$ is __________________ if every element of its range corresponds to exactly one element of the domain.

(b) A function $f$ from set $A$ to set $B$ is __________________ if for every element $b$ in $B$, there exists an element in $A$ such that $f(a) = b$.

(c) A function $f$ is __________________ if it is one-to-one and onto.

(2) Let $A$ and $B$ be finite sets of cardinalities $|A|$ and $|B|$, respectively. Fill in the blanks with $\geq$, $=$, or $\leq$.

(a) There exists a one-to-one function $g : A \rightarrow B$ if and only if $|A|$______$|B|$.

(b) There exists an onto function $g : A \rightarrow B$ if and only if $|A|$______$|B|$.

(c) Assuming that (a) and (b) are true, prove that there exists a bijective function $g : A \rightarrow B$ if and only if $|A| = |B|$.

We say that two sets $A$ and $B$ are isomorphic if there is a bijection $f : A \rightarrow B$. 
(3) In your own words, explain what it means for a set to be “infinite”?

(4) Consider the set of natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots \} \) and the set of positive odd numbers \( \mathbb{N}_{\text{odd}} = \{1, 3, 5, \ldots \} \). Notice that both sets are not finite. Do you think that \( \mathbb{N} \) contains more elements than \( \mathbb{N}_{\text{odd}} \)?
Subsets and Proper Subsets

**Definition 1.** A set $A$ is a subset of another set $B$ if all elements of the set $A$ are elements of the set $B$.

For example, in the picture below, the set $A$ is a subset of set $B$ because all three elements of $A$ are also elements of $B$. Similarly, the set $C$ is a subset of set $D$ because all five elements of $C$ are also elements of $D$.

![Subset Diagram]

**Problem 1.** Is $X$ a subset of itself?

**Problem 2.** True/False: The empty set $\emptyset$ a subset of any set.

**Definition 2.** A subset of a set $X$ is called a proper subset if it is a subset which is not $\emptyset$ or $X$.

In the example above, the set $A$ is a proper subset of set $B$ since there are two elements of $B$ that are not elements of $A$. However the set $C$ is not a proper subset of set $D$ because all five elements of $C$ are also elements of $D$.

**Problem 3.** Describe a proper subset of the students in your Math Circle class.
Problem 4. List two proper subsets of the set of natural numbers $\mathbb{N}$. Make one of your subsets finite and another “infinite”.

Problem 5. Explain why the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ is a proper subset of the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Problem 6. Let $A$ and $B$ be finite sets. Prove that if $A$ is a proper subset of $B$ then $B$ is not a subset of $A$. 
Problem 7. Let $A$ and $B$ be finite sets. Prove that if $A$ is a subset of $B$ then $|A| \leq |B|$.

Problem 8. Let $A$ be a proper subset of $B$. Write the inequality relating $|A|$ and $|B|$.

Problem 9. Let $B$ be a finite set and $A$ be a proper subset of $B$. Is it possible that there is a bijection between $A$ and $B$?
Finite and Infinite Sets

So far, we have been talking about finite and infinite sets without having defined what they mean. As seen from the warm up, this is difficult.

**Definition 3.** We say that a set is **finite** if it is isomorphic to \( \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \).

For example, the set \( X = \{\text{Ivy, Luke, Dani, Nikola, Benjamin}\} \) is finite because it is isomorphic to \( \{1, 2, \ldots, n\} \) where \( n = 5 \). The isomorphism is given by the bijection \( f : X \to \{1, 2, \ldots, 5\} \) where \( f \) is defined by the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \mathbb{N} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivy</td>
<td>1</td>
</tr>
<tr>
<td>Luke</td>
<td>2</td>
</tr>
<tr>
<td>Dani</td>
<td>3</td>
</tr>
<tr>
<td>Nikola</td>
<td>4</td>
</tr>
<tr>
<td>Benjamin</td>
<td>5</td>
</tr>
</tbody>
</table>

**Problem 10.** Show that \( X = \{\text{red, green, blue, yellow, orange}\} \) is finite.
Intuitively, we can say that a set $A$ is an infinite set if we can remove elements of $A$ without changing the “size” of $A$.

We can give a precise definition of the above intuition as the following:

**Definition 4.** A set is an **infinite set** if it is isomorphic to some proper subset of itself.

For example, we can show that the set of integers is infinite by constructing a bijection from $\mathbb{Z}$ to its proper subset, $\mathbb{N}$.

We do this by assigning to each element element in $\mathbb{Z}$ an element in $\mathbb{N}$.

<table>
<thead>
<tr>
<th>$\mathbb{Z}$</th>
<th>0</th>
<th>1</th>
<th>$-1$</th>
<th>2</th>
<th>$-2$</th>
<th>3</th>
<th>$-3$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{N}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>...</td>
</tr>
</tbody>
</table>

The images of the positive elements of $\mathbb{Z}$ are the even natural numbers. The images of the non-positive elements of $\mathbb{Z}$ are the odd natural numbers. Therefore, the function $f : \mathbb{Z} \rightarrow \mathbb{N}$ is given by the formula

$$f(z) = \begin{cases} 2z & \text{if } z \text{ is positive} \\ -2z + 1 & \text{otherwise} \end{cases}.$$  

We have already shown that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$. What remains is to show that $f$ is a bijection between the sets.

**Problem 11.** Prove that $f$ is a bijection.
Problem 12. Plotting the graph of $f : \mathbb{Z} \to \mathbb{N}$

(1) Plot the line $y = 2x$ and $y = -2x + 1$ on the $xy$-plane.
(2) Use these lines to show the graph of $f : \mathbb{Z} \to \mathbb{N}$ by drawing the integer points on the graph of the function.
Problem 13. Benjamin is trying to prove that $\mathbb{Z}$ is an infinite set by using the definition above. He claims he has proven that $\mathbb{Z}$ is an infinite set because he has constructed a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ given by the formula
\[ g(z) = |z| + 1. \]
Is Benjamin correct? Explain why or why not.

Problem 14. Dani is also trying to prove that $\mathbb{Z}$ is an infinite set by using the definition above. She claims she has proven that $\mathbb{Z}$ is an infinite set because she has constructed a function $h : \mathbb{Z} \rightarrow X$ given by the formula
\[ g(z) = \begin{cases} 
  z + 1 & \text{if } z \geq 0 \\
  z & \text{if } z < 0
\end{cases} \]
where $X = \{..., -2, -1, 1, 2, ...\}$.
Is Dani correct? Explain why or why not.
Problem 15. Let $N_{\text{even}}$ be the set of non-negative even numbers.

(1) List the elements of $N_{\text{even}}$ in set notation.

(2) Is $N_{\text{even}}$ a proper subset of $\mathbb{N}$? Why or why not?

(3) Prove that $\mathbb{N}$ is infinite by constructing a bijection from $\mathbb{N}$ to $N_{\text{even}}$. 
Cardinalities of Infinite Sets

While it is easy to compare the cardinalities of finite sets by simply counting the number of elements in each set, we cannot do the same for when the sets are infinite. As shown from the warm up, we can compare the cardinalities of finite sets by examining the properties of the functions that map one set to another. We can compare the cardinalities of infinite sets in a similar manner:

Let $A$ and $B$ be infinite sets of cardinalities $|A|$ and $|B|$, respectively.

- If there exists a bijective function $g : A \to B$, then $|A| = |B|$.

Problem 16. Show that the set of natural numbers $\mathbb{N}$ and the set of odd numbers $\mathbb{N}_{\text{odd}}$ are isomorphic.

In other words, find a bijection from the set of natural numbers $\mathbb{N}$ and the set of odd numbers $\mathbb{N}_{\text{odd}}$ to conclude that $|\mathbb{N}| = |\mathbb{N}_{\text{odd}}|$.

Was your hypothesis from warm up problem (4) correct?
Problem 17. (Challenge) Prove that the set of natural numbers $\mathbb{N}$ and the set of positive rational numbers $\mathbb{Q}^+ = \{ \frac{a}{b} \mid a, b \in \mathbb{N}, a > 0, b > 0 \}$ have the same cardinalities.

Hint: Do not try to write a function from $\mathbb{N}$ to $\mathbb{Q}^+$. Instead, picture $\frac{a}{b}$ as the points $(a, b)$ on a coordinate plane with $a, b > 0$. Find a way to number all the pairs which are in simplest form.

Problem 18. Do you think that all infinite sets have the same cardinality? We will see if you are right in the future handouts.