Theory of cubic equations

Problem 1  Find the coefficients $a$, $b$, $c$, and $d$ of the cubic function $y = ax^3 + bx^2 + cx + d$ given by the following graph.
A generic cubic equation

$$az^3 + bz^2 + cz + d = 0$$  \quad (1)

with real coefficients $a \neq 0$, $b$, $c$, and $d$ is equivalent to the equation

$$z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = 0.$$  \quad (2)

**Problem 2** Find the variable change $z = x - x_0$ reducing the equation \[(2)\] to the **depressed cubic form**

$$x^3 + px + q = 0.$$  \quad (3)
Problem 3 Use the expansion
\[(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = 3uv(u + v) + u^3 + v^3 \quad (4)\]
and the Vieta formulas for a quadratic equation to find a root of (3). Hint: rewrite (3) as \[x^3 = -px - q\] and compare the latter to (4).

The above method of solving a cubic equation was discovered by an Italian mathematician Scipione del Ferro (1465 - 1526), independently rediscovered by an Italian engineer Tartaglia (1500-1557), and published, with attribution to del Ferro, in 1545 by Gerolamo Cardano (1501-1576) in his book *Ars Magna*. In full accordance with Arnold’s Law, the formula for the root is known as the *Cardano formula*. 
Problem 4 Find a real root of the equation $x^3 + 6x - 2 = 0$.

Problem 5 Find a real root of the equation $x^3 + 6x^2 + 9x - 2 = 0$. 
Long division of polynomials

**Example 1**  
*Divide* \( x^3 + 3x^2 + 5x - 4 \) *by* \( x - 1 \).

**Step 1:** multiply \( x - 1 \) by a monomial of the form \( ax^n \) so that the leading term of the product equals the leading term of \( x^3 + 3x^2 + 5x - 4 \). In our case, \( ax^n = x^2 \). Subtract the product, \( x^2(x - 1) = x^3 - x^2 \), from the original polynomial.

\[
\begin{array}{c|cc}
   & x^2 & \\
\hline
   x - 1 & x^3 & + 3x^2 & + 5x & - 4 \\
   & x^3 & - x^2 & \\
\hline
   & & 4x^2 & + 5x & - 4
\end{array}
\]

**Step 2:** repeat step 1 for the polynomials \( 4x^2 + 5x - 4 \) and \( x - 1 \).

\[
\begin{array}{c|cc}
   & x^2 & + 4x & \\
\hline
   x - 1 & x^3 & + 3x^2 & + 5x & - 4 \\
   & x^3 & - x^2 & \\
\hline
   & & 4x^2 & + 5x & - 4 \\
   & & 4x^2 & - 4x & \\
\hline
   & & 9x & - 4
\end{array}
\]

**Problem 6** Perform step 3 and finish the division in Example above.
Problem 7 Divide $3x^5 - 2x^3 + 10x^2 - x + 11$ by $x^2 + x + 1$. 
Problem 8 Without using the Cardano formula, find all the three roots of the equation \( x^3 - 5x - 2 = 0 \).
Problem 9  Use the Cardano formula to find a root of the equation $x^3 - 5x - 2 = 0$ from Problem 8. Which of the three roots found on the previous page is this one equal to? Hint: check them out one by one.
Complex numbers

The imaginary unit $i$ is defined as one of the two solutions of the equation $x^2 + 1 = 0$. The other one is $-i$.

**Problem 10** Evaluate $i^{2017}$.

The set $\mathbb{C}$ of complex numbers is defined as the set of all the numbers of the form $z = a + ib$ where $a$ and $b$ are real numbers. The real part of the complex number $z$ is $\mathcal{R}(z) = a$. The imaginary part of the complex number $z$ is $\mathcal{I}(z) = b$. Complex numbers are constructed to have the same algebraic properties as real numbers. In particular, multiplication by $i$ is commutative for any real number, $ib = bi$. Addition of real and imaginary parts is also commutative, $a + ib = ib + a$.

**Problem 11** Given $z = 1 - 3i + \sqrt{2}$, find $\mathcal{R}(z)$ and $\mathcal{I}(z)$.

Complex numbers were invented by Gerolamo Cardano in an attempt to resolve the paradox similar to the one we have encountered comparing the solutions of the cubic equation in Problems 8 and 9. Complex numbers are very important. For example, complex numbers form the bedrock of quantum mechanics.
Problem 12 Prove that the sum of any two complex numbers, \( v = a + ib \) and \( w = c + id \), is also a complex number.

Note that addition of complex numbers is commutative, \( v + w = w + v \).

Problem 13 Given \( p = 7 - i \), \( q = 2 + 2i \), and \( r = -5 + i\sqrt{3} \), find \( p + q + r \).

Problem 14 For the numbers \( p \), \( q \), and \( r \) from Problem 13, find \( p^2 + q^2 + r^2 \).

Problem 15 Is any real number a complex number? Why or why not?
Problem 16 Prove that zero is the only neutral element with respect to addition of complex numbers. In other words, \( z + n = z \) for any complex number \( z \) and some complex number \( n \) if and only if \( n = 0 \).

Problem 17 Prove that for any complex number \( z \) there exists the opposite complex number, \(-z\), such that \( z + (-z) = 0 \). What are \( \Re(-z) \) and \( \Im(-z) \)?

Problem 18 Prove that multiplication of complex numbers is commutative. In other words, prove that for any \( v = a + ib \) and \( w = c + id \), \( vw = wv \). What are \( \Re(vw) \) and \( \Im(vw) \)?
Problem 19  For the numbers $p = 7 - i$, $q = 2 + 2i$, and $r = -5 + i\sqrt{3}$ from Problem 13, find the number $pqr$.

Problem 20  Prove that 1 is the only neutral element with respect to multiplication of complex numbers. In other words, $zn = z$ for any complex number $z$ and some complex number $n$ if and only if $n = 1$. 
Problem 21 Prove that for any complex number \( z = a + ib \neq 0 \) there exists the inverse complex number, \( z^{-1} \), such that \( z^{-1}z = 1 \). Hint: \( a + ib \neq 0 \iff a^2 + b^2 \neq 0 \). What are \( \Re(z^{-1}) \) and \( \Im(z^{-1}) \)?

Problem 22 For the numbers \( p = 7 - i \), \( q = 2 + 2i \), and \( r = -5 + i\sqrt{3} \) from Problem 13, find the numbers \( p^{-1} \), \( q^{-1} \), and \( r^{-1} \).
Let \( z = a + ib \). The number \( \bar{z} = a - ib \) is called conjugate to \( z \).

**Problem 23** For any two complex numbers \( v \) and \( w \), prove that \( \bar{v + w} = \bar{v} + \bar{w} \).

**Problem 24** For any two complex numbers \( v \) and \( w \), prove that \( \bar{vw} = \bar{v} \bar{w} \).

**Problem 25** Prove that \( z \in \mathbb{R} \iff z = \bar{z} \).

**Problem 26** Prove that for \( z = a + ib \), \( z\bar{z} = a^2 + b^2 \).

Problem [26] justifies the following definition.

\[
|z| = \sqrt{z\bar{z}}
\]  
(5)
Problem 27  For the numbers \( p = 7 - i \), \( q = 2 + 2i \), and \( r = -5 + i\sqrt{3} \) from Problem 13, find \( |p| \), \( |q| \), and \( |r| \).

Problem 28  Find all the (complex) roots of the equation

\[ x^2 + 2x + 3 = 0. \]