The following is known as a \textit{discriminant} of the quadratic equation \( ax^2 + bx + c = 0, \ a \neq 0 \).

\[ D = b^2 - 4ac \quad (1) \]

**Theorem 1** If \( D < 0 \), then the quadratic equation \( ax^2 + bx + c = 0 \) with real coefficients \( a \neq 0, \ b, \) and \( c \) has no real roots. If \( D \geq 0 \), then the following is the formula for the roots of the equation.

\[ x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad (2) \]

**Problem 1** Prove Theorem [1]
Problem 2 \hspace{1em} \textit{Find all the real solutions of the equation} \\
\sqrt{x - 2} = x - 4.

Problem 3 \hspace{1em} \textit{Find all the real solutions of the equation} \\
7 \left( x + \frac{1}{x} \right) - 2 \left( x^2 + \frac{1}{x^2} \right) = 9.
Problem 4 Sketch the graph of the function $y = ax^2 + bx + c$, given the following information: $a > 0$, $b > 0$, $D < 0$.

Is the coefficient $c$ positive, negative, or zero? Why?
Problem 5 Find all the real solutions of the equation
\[2x^2 + 6 - 2\sqrt{2x^2} - 3x + 2 = 3x + 3.\]

Problem 6 Find all the real solutions of the equation
\[3\sqrt{x + a} + 3\sqrt{x + a + 1} + 3\sqrt{x + a + 2} = 0.\]
Vieta Formulas

**Theorem 2** Let $x_1$ and $x_2$ be the roots of the quadratic equation $ax^2 + bx + c$, $a \neq 0$. Then $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$.

**Problem 7** Prove Theorem 2.

**Problem 8** Write down a quadratic equation that has the roots $x_1 = 3$ and $x_2 = -4$. 
Problem 9  Generalize Vieta formulas to a cubic equation \( ax^3 + bx^2 + cx + d, \ a \neq 0. \)

Problem 10  Write down a cubic equation that has the roots \( x_1 = 1, \ x_2 = 2, \ and \ x_3 = 3. \)
Problem 11  Without solving the equation $ax^2 + bx + c = 0$, find the sum of the squares of its roots provided that $a \neq 0$ and $D \geq 0$.

Problem 12  Find all the prime numbers $p$ and $q$ such that the equation $x^2 - px - q = 0$ has a solution that is a prime number.
A function $f(x)$ is called convex if for any $x_1$ and $x_2$ in its domain and for any $0 < \alpha < 1$,

$$f(\alpha x_1 + (1 - \alpha) x_2) < \alpha f(x_1) + (1 - \alpha) f(x_2).$$

(3)

**Problem 13** Give a geometric interpretation to formula (3).

**Problem 14** Prove that for a linear function $f(x) = bx + c$, $f(\alpha x_1 + (1 - \alpha) x_2) = \alpha f(x_1) + (1 - \alpha) f(x_2)$ for any value of the parameter $\alpha$. 
Problem 15  Prove that \( f(x) = ax^2 + bx + c \) is convex for \( a > 0 \).

The value \( \hat{x} \) is called a minimum of a function \( f(x) \) if \( f(\hat{x}) \leq f(x) \) for every \( x \) in the function’s domain.

Problem 16 Sketch the graph of a function having two minima.
Problem 17  The function $f(x)$ is convex. Prove that it can have at most one minimum.

Problem 18  Find the minimum of the function $f(x) = ax^2 + bx + c$, $a > 0$. Prove that it is indeed a minimum. What is the value of the function at the point?
Problem 19 Find the minimum of the function
\[ f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2. \]

Problem 20 A straight line in the plain is given by the equation
\[ ax + by + c = 0. \] Find the distance from the point \((x_0, y_0)\) to the line.
Problem 21  Prove that for $x > 0$, $x + \frac{1}{x} \geq 2$.

Problem 22  Given $x + y + z = 1$, $x > 0$, $y > 0$, and $z > 0$, prove that 

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$